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EE 16B Final
Spring 2018

Name: JAIJEET ROYCHOWDHURY

(after the exam begins add your SID# in the top right corner of each page)

Discussion Section and TA:______________________________
Discussion Section and TA:______________________________
Lab Section and TA:____________________________________

Name of left neighbor:__________________________________
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Instructions:

Show your work. An answer without explanation is not acceptable and does not guarantee any credit.

Only the front pages will be scanned and graded. Back pages won't be scanned; you can use them as scratch paper.

Do not remove pages, as this disrupts the scanning. If needed, cross out any parts that you don't want us to grade.

PROBLEM	MAX
1	17
2	8
3	15
4	10
5	10
6	5
7	15
8	20
TOTAL	100

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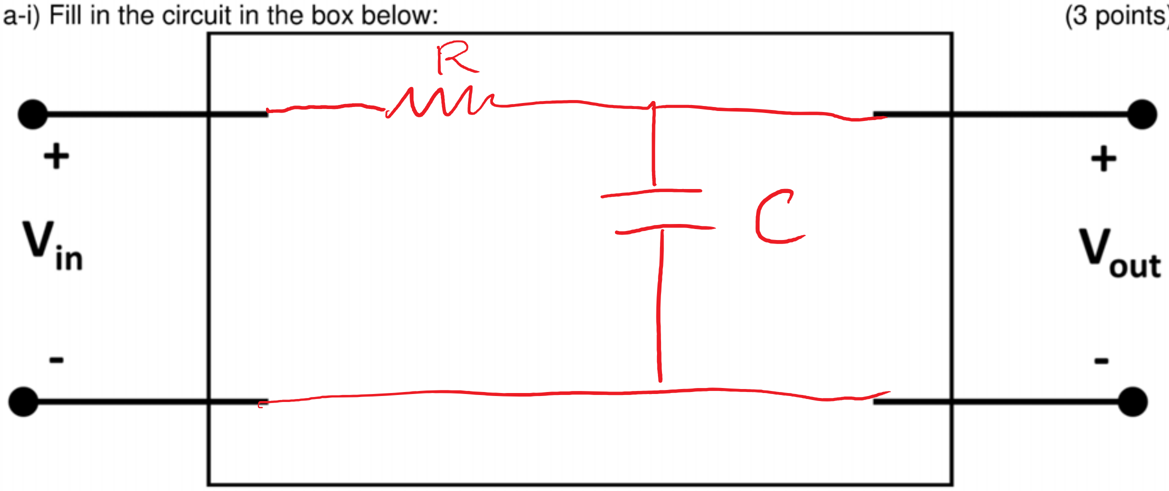
Problem 1 (17 points) **[CAN CLOBBER MT1]**

Consider the box below. It has one input and one output. **Inside the box**, using any number of op amps, resistors, capacitors, inductors, and/or transistors, **draw the simplest** circuit that has the following properties:

- it is a voltage low pass filter
- the cutoff frequency, f_c, of the low pass filter is 10 kHz. Recall that $\omega c = 2*\pi*f_c$.
- the slope of the magnitude of the transfer function, $H(\omega) = \frac{V_{out}}{V_{in}}$ as $\omega \to \infty$ is -20 dB/decade
- the voltage gain at $\omega \to 0$ is 1

1a-i) Fill in the circuit in the box below: (3 points)



1a-ii) Provide numerical values for all components that need numerical values below. (2 points)

Values:

$$\omega_c = \frac{1}{RC} = 2\pi \cdot f_c$$

$$RC = \frac{1}{2\pi \cdot 10^4}$$ ← any combo of R, C that meets this eqn is ok

$R = 1k\Omega$ $C = \frac{1}{2\pi} \cdot 10^{-7}$ F

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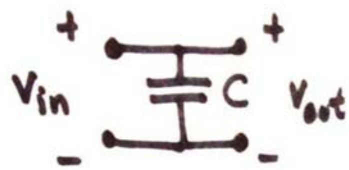
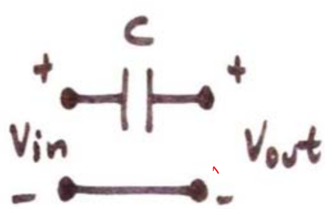
1b) Consider the circuits below. For each circuit, provide the voltage transfer functions specified. All inputs are sinusoidal in the time domain. (1.5 points each = 12 points total)



$$H(\omega) = \frac{V_{out}}{V_{in}} = 1$$

$V_{in} = V_{out}$



$$H(\omega) = \frac{V_{out}}{V_{in}} = 1$$

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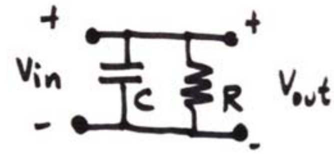
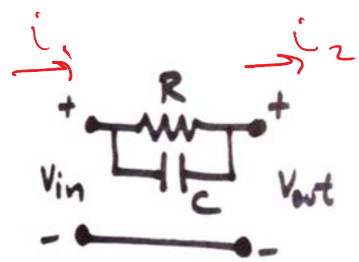


$$H(\omega) = \frac{V_{out}}{V_{in}} = 1 \qquad V_{out} = V_{in}$$

no loop for i_1 to flow



$i_1 = i_2 = 0$

$\Rightarrow V_{in} - V_{out} = 0$

$V_{in} = V_{out}$

$$H(\omega) = \frac{V_{out}}{V_{in}} = 1$$

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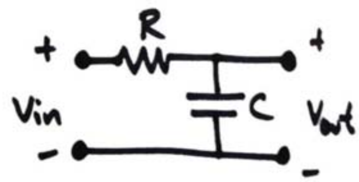
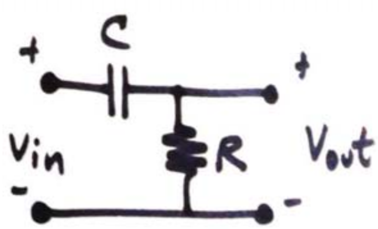


$$H(\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1+j\omega CR}$$

Volt. divider

$$\frac{1/j\omega C}{1/j\omega C + R} = \frac{1}{1+j\omega CR}$$



$$H(\omega) = \frac{V_{out}}{V_{in}} = \frac{j\omega CR}{1+j\omega CR}$$

Volt. divider

$$\frac{R}{1/j\omega C + R} = \frac{j\omega CR}{1+j\omega CR}$$

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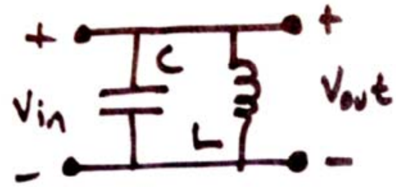
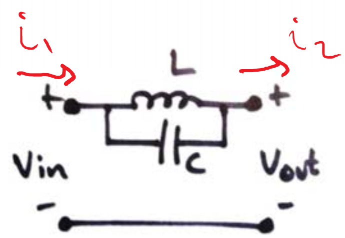


$$H(\omega) = \frac{V_{out}}{V_{in}} = \quad 1$$

$V_{out} = V_{in}$



no loop for i_1 to flow

$i_1 = i_2 = 0$

$$H(\omega) = \frac{V_{out}}{V_{in}} = \quad 1$$

$V_{in} = V_{out}$

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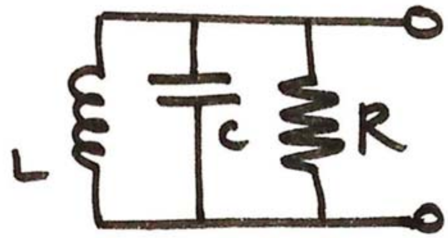
Problem 2 (8 points) **[CAN CLOBBER MT1]**

Consider the circuit below. C = 100 nF; L = 15 µH; R = 100 Ω. Wherever relevant, the variable $i_L(t)$ is the time domain current flowing through the inductor. For the circuit below, assume $i_L(t=0)$ = 3 mA.



2a) Is the concept of a quality factor, Q, meaningful for this circuit? (Circle **Yes or No** below). If Yes, what is the value? (3 points)

YES **NO**

Q =

Method 1:

$$Q = \frac{\omega_0}{BW} \qquad \omega_0 = 1/\sqrt{LC} \qquad BW = \frac{1}{RC}$$

Parallel RLC BW

$$Q = \omega_0 RC = \frac{RC}{\sqrt{LC}}$$

$$Q = \frac{100 \cdot 100 \cdot 10^{-9}}{\sqrt{15 \cdot 10^{-6} \cdot 100 \cdot 10^{-9}}} = \frac{10^{-5}}{\sqrt{15 \cdot 10^{-13}}}$$

$$Q = \frac{\sqrt{10}}{\sqrt{15}} \cdot 10 = \sqrt{\frac{2}{3}} \cdot 10$$

Method 2: $Q = 2\pi \left.\frac{U_{stored}}{U_{dis.}}\right|_{\omega=\omega_0}$

$$U_{dis} = \int_0^{2\pi/\omega_0} \frac{v(t)^2}{R} = \int_0^{2\pi/\omega_0} \frac{V_0 \cos^2(\omega_0 t)}{R} = \frac{\pi V_0^2}{\omega_0 R}$$

$$U_{stored} = U_L + U_C$$

$$U_L = \frac{1}{2} L i_L(t)^2 = \frac{1}{2} L \left(\int \frac{V_0 \cos(\omega_0 t)}{L}\right)^2 = \frac{V_0^2 \sin^2(\omega_0 t)}{2L\omega_0^2} = \tfrac{1}{2} C V_0^2 \sin^2(\omega_0 t)$$

$\omega_0 = \frac{1}{\sqrt{LC}}$

$$U_C = \frac{1}{2} C v_C(t)^2 = \frac{1}{2} C V_0^2 \cos^2(\omega_0 t)$$

$$U_{stored} = \frac{1}{2} C V_0^2 \underbrace{\left(\cos^2(\omega_0 t) + \sin^2(\omega_0 t)\right)}_{1} = \frac{1}{2} C V_0^2$$

$$Q = 2\pi \cdot \frac{\tfrac{1}{2} C V_0^2}{\pi V_0^2 / \omega_0 R} = \omega_0 RC = \boxed{\sqrt{\frac{2}{3}} \cdot 10}$$

Method 1: $Q > 1 \Rightarrow$ oscillations ✓

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2b) In the time domain, is the natural response of this circuit such the voltage across the resistor will oscillate with time for t > 0? (2 points)

$i_L = -i_c - i_R \qquad V_c = V_L = V_R$

$i_L = -C\frac{dV_c}{dt} - \frac{V_c}{R} \qquad i_R = \frac{V_R}{R} = \frac{V_c}{R}$

$i_c = C\frac{dV_c}{dt}$

YES **NO**

$L\frac{di_L}{dt} = V_L = V_c$

$L\frac{d}{dt}\left(-C\frac{dV_c}{dt} - \frac{V_c}{R}\right) = V_c \Rightarrow -LC\frac{d^2V_c}{dt^2} - \frac{dV_c}{dt}\frac{L}{R} = V_c$

$$\frac{d^2V_c}{dt} + \frac{dV_c}{dt}\cdot\underbrace{\frac{1}{RC}}_{2\alpha} + V_c\cdot\underbrace{\frac{1}{LC}}_{w_0^2} = 0$$

$\alpha = \frac{1}{2}\cdot\frac{1}{100\cdot100\cdot10^{-9}} = 5\cdot10^4 \qquad w_0 = \frac{1}{\sqrt{15\cdot10^{-6}\cdot100\cdot10^{-9}}} = \sqrt{\frac{2}{3}}\cdot10^6$

2c) If yes, what would be the angular frequency of that oscillation? (3 points)

$\boxed{\alpha < w_0 \Rightarrow \text{oscillates}}$

$\omega =$

$$w_D = \sqrt{w_0^2 - \alpha^2} = \boxed{\sqrt{\frac{2}{3}\cdot10^{12} - 25\cdot10^8}\ \frac{rad}{s}}$$

$$= \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$$

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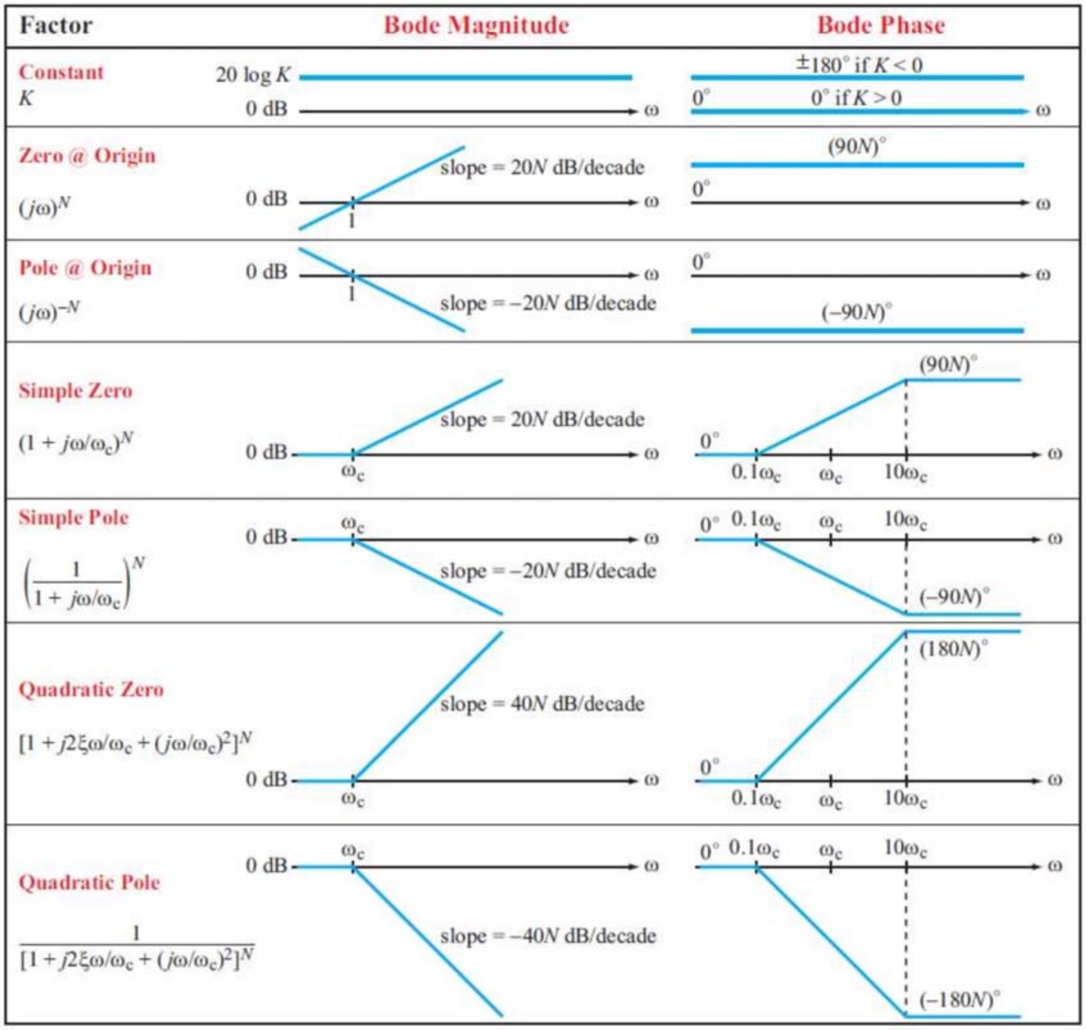

Factor
Bode Magnitude
Bode Phase
Constant
K
20 log K
0 dB
ω
±180° if K < 0
0°
0° if K > 0
ω
Zero @ Origin
(jω)^N
slope = 20N dB/decade
0 dB
1
ω
(90N)°
0°
ω
Pole @ Origin
(jω)^−N
0 dB
1
ω
slope = −20N dB/decade
0°
ω
(−90N)°
Simple Zero
(1 + jω/ωc)^N
slope = 20N dB/decade
0 dB
ωc
ω
(90N)°
0°
0.1ωc
ωc
10ωc
ω
Simple Pole
(1/(1 + jω/ωc))^N
0 dB
ωc
ω
slope = −20N dB/decade
0° 0.1ωc
ωc
10ωc
ω
(−90N)°
Quadratic Zero
[1 + j2ξω/ωc + (jω/ωc)²]^N
slope = 40N dB/decade
0 dB
ωc
ω
(180N)°
0°
0.1ωc
ωc
10ωc
ω
Quadratic Pole
1/[1 + j2ξω/ωc + (jω/ωc)²]^N
0 dB
ωc
ω
slope = −40N dB/decade
0° 0.1ωc
ωc
10ωc
ω
(−180N)°

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Problem 3 (15 points) State-Space Equations, Feedback and Stability **[CAN CLOBBER MT2]**

(This problem brings out a key issue encountered when designing op-amps - take EE140 to learn more.)

It is common in electronic system design to use op-amps in negative feedback configurations, as shown below.

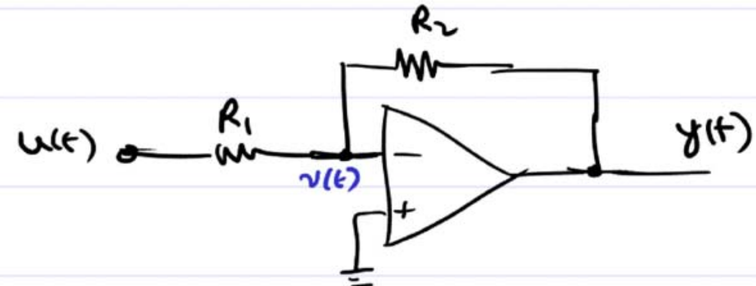

Fig. 1: Op-amp in negative feedback configuration. u(t), v(t) and y(t) all refer to the voltages at the corresponding nodes.

In idealized operation, the voltage v(t) at the "-" terminal of the op-amp is assumed to be at "virtual ground". No current enters or leaves the "-" terminal, hence applying KCL at that terminal results in the I/O relationship $y(t) = -\frac{R_2}{R_1}u(t)$.

A more realistic analysis, however, would represent the internals of the op-amp as shown below:

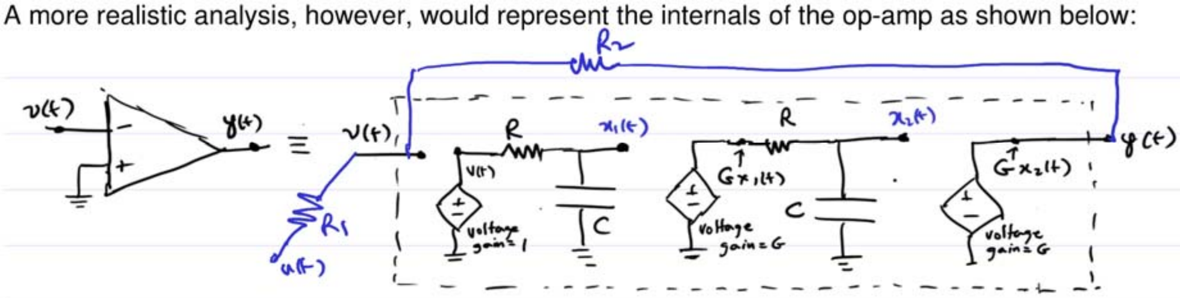

Fig. 2: More realistic model of op-amp internals. Note the internal node voltages $x_1(t)$ and $x_2(t)$.

NOTE: As you go through this question, you may find that your derivations and results challenge previous (simplistic) notions you may have about op-amps. That's what this question is meant to do.

(Turn to the next page for the questions.)

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3a) Using $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, the op-amp's internal circuitry in Fig. 2 can be expressed in state space form as

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}v(t), \text{ with } y(t) = \vec{c}^T\vec{x}(t).$$

Write out expressions (involving R, C, and G) for A, $\vec{b}$ and $\vec{c}^T$. (3 points)

$$\text{KCL @ } x_1: \quad C\frac{dx_1}{dt} = \frac{v(t) - x_1(t)}{R} \implies \frac{dx_1}{dt} = \frac{v(t) - x_1(t)}{RC}$$

$$\text{KCL @ } x_2: \quad C\frac{dx_2}{dt} = \frac{Gx_1(t) - x_2(t)}{R} \implies \frac{dx_2}{dt} = \frac{Gx_1(t) - x_2(t)}{RC}$$

$$\implies \frac{d}{dt}\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\vec{x}(t)} = \underbrace{\begin{bmatrix} -1/RC & 0 \\ G/RC & -1/RC \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix}}_{\underline{\vec{b}}} v(t)$$

$$\longrightarrow y(t) = G\,x_2(t) = \begin{bmatrix} 0 & G \end{bmatrix} \vec{x}(t)$$

$$\implies \underline{\vec{c}^T = \begin{bmatrix} 0 & G \end{bmatrix}}$$

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3b) Find expressions for the eigenvalues of A and indicate if the op-amp model can be i) stable, ii) marginally stable and iii) unstable, explaining each answer. Note that R, C and G can only take values > 0, i.e. they cannot be zero or negative. (3 points)

$$A = \begin{bmatrix} -1/RC & 0 \\ G/RC & -1/RC \end{bmatrix} ; \quad (A - \lambda I) = \begin{bmatrix} -1/RC - \lambda & 0 \\ G/RC & -1/RC - \lambda \end{bmatrix} ;$$

$\rightarrow \det(A - \lambda I) = (\lambda + 1/RC)^2$

$\rightarrow$ Eigenvalues: $\underline{\lambda_1, \lambda_2 = -1/RC}$ (repeated)

i) Since R&C are >0, $-1/RC < 0$ and real $\Rightarrow$ STABLE for all RC

ii, iii) NOT POSSIBLE

3c) If we use the op-amp model of Fig. 2 in the negative-feedback circuit of Fig. 1, the overall closed-loop circuit can also be written in state-space form as

$$\frac{d}{dt}\vec{x}(t) = A_f\vec{x}(t) + \vec{b}_f u(t), \text{ with } y(t) = \vec{c}^T\vec{x}(t).$$

Write out expressions for A_f and $\vec{b}_f$ in terms of R, C, G, R_1 and R_2. Note that $\vec{x}(t)$ and $\vec{c}^T$ are the same as in 3a), and that the **input is now $u(t)$ and not $v(t)$.** (5 points)

→ FROM THE FEEDBACK (KCL @ $v(t)$), WE HAVE $\dfrac{u(t)-v(t)}{R_1} = \dfrac{v(t)-y(t)}{R_2} = \dfrac{v(t)-Gx_2(t)}{R_2}$

$$\Rightarrow v(t)\left[\frac{1}{R_1}+\frac{1}{R_2}\right] = \frac{u(t)}{R_1} + \frac{Gx_2(t)}{R_2}$$

$$\Rightarrow v(t) = \frac{GR_1\, x_2(t) + R_2 u(t)}{R_1+R_2}$$

→ Hence 3a) becomes:

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\vec{x}(t)} = \underbrace{\begin{bmatrix} -1/RC & 0 \\ G/RC & -1/RC \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix}}_{\vec{b}}\left[\frac{GR_1}{R_1+R_2}x_2(t) + \frac{R_2}{R_1+R_2}u(t)\right]$$

$$= \underbrace{\begin{bmatrix} -1/RC & \frac{GR_1}{RC(R_1+R_2)} \\ \frac{G}{RC} & -1/RC \end{bmatrix}}_{\underline{A_f}}\vec{x}(t) + \underbrace{\begin{bmatrix} \frac{R_2}{RC(R_1+R_2)} \\ 0 \end{bmatrix}}_{\underline{\vec{b}_f}} u(t)$$

3d) Can the closed-loop system of 3c) be i) stable, ii) marginally stable and iii) unstable? For **each one that is possible,** write out a condition (i.e., an equation or inequality involving R, C, G, R_1 and R_2) that results in that stability property. (4 points)

$$A_f = \underbrace{\left[\begin{array}{c|c} -1/RC & \frac{GR_1}{RC(R_1+R_2)} \\ \hline \frac{G}{RC} & -1/RC \end{array}\right]}\ ; \quad \text{char. poly: } \left(\lambda + 1/RC\right)^2 - \frac{G^2}{(RC)^2}\frac{R_1}{R_1+R_2}$$

$$\longrightarrow \text{Eigenvalues: } \left(\lambda + 1/RC\right) = \pm \frac{G}{RC}\sqrt{\frac{R_1}{R_1+R_2}}$$

$$\Longrightarrow \lambda_1, \lambda_2 = \frac{-1}{RC} \pm \frac{G}{RC}\sqrt{\frac{R_1}{R_1+R_2}}$$

(ii) (marginally stable): $\lambda_1 = 0 \Longrightarrow \frac{G}{RC}\sqrt{\frac{R_1}{R_1+R_2}} = \frac{1}{RC}$

$$\Longrightarrow \underline{G = \sqrt{\frac{R_1+R_2}{R_1}}}$$ ← CONDITION FOR MARGINAL STABILITY: POSSIBLE.

(i) (stable) $\frac{G}{RC}\sqrt{\frac{R_1}{R_1+R_2}} < \frac{1}{RC} \Rightarrow \lambda_1, \lambda_2$ both < 0 (and real)

$$\Longrightarrow \underline{G < \sqrt{\frac{R_1+R_2}{R_1}}}$$ ← CONDITION FOR STABILITY: POSSIBLE

(iii) (unstable) $\frac{G}{RC}\sqrt{\frac{R_1}{R_1+R_2}} > \frac{1}{RC} \Rightarrow \lambda_1 > 0$ (and real)

$$\Longrightarrow \underline{G > \sqrt{\frac{R_1+R_2}{R_1}}}$$ ← CONDITION FOR INSTABILITY: POSSIBLE

Problem 4 (10 points) Observability and Observers

Consider the discrete-time state space system

$$\underbrace{\begin{bmatrix} p[t+1] \\ v[t+1] \end{bmatrix}}_{\vec{x}[t+1]} = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p[t] \\ v[t] \end{bmatrix}}_{\vec{x}[t]} + \underbrace{\begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}}_{\vec{b}} u[t]$$, where T is a time interval (real number > 0).

(This system is exactly the same as the one for the car with piecewise-constant (PWC) acceleration covered in class. $u[t]$ is the PWC acceleration applied to the car, $p[t]$ is its position at time tT, and $v[t]$ is its velocity at time tT.)

We augment the above equation with an output equation, i.e.,

y[t] = p[t].

4a) Find the eigenvalues of A. (1 point)

$\longrightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda_{1,2} = 1$ (repeated eigenvalues)

4b) Suppose u[t] = 0 for all t. Derive expressions for $\vec{x}[t]$ if (i) $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and (ii) $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. (1 point)

(i) $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\rightarrow \vec{x}[1] = A\vec{x}[0] = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x}[0]$

$\rightarrow \underline{\vec{x}[t] = \vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \; \forall t$

(ii) $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x}[1] = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T \\ 1 \end{bmatrix}$

$\vec{x}[2] = \begin{bmatrix} T \\ 0 \end{bmatrix} + \begin{bmatrix} T \\ 1 \end{bmatrix} = \begin{bmatrix} 2T \\ 1 \end{bmatrix}$

$\vec{x}[3] = \begin{bmatrix} 2T \\ 0 \end{bmatrix} + \begin{bmatrix} T \\ 1 \end{bmatrix} = \begin{bmatrix} 3T \\ 1 \end{bmatrix}$

$\cdots \; \underline{\vec{x}[t] = \begin{bmatrix} tT \\ 1 \end{bmatrix}} \longleftarrow$

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4c) Is the system observable? Justify/derive your answer. (1 point)

$\vec{c}^T = [1 \quad 0]$: Observability matrix $= \begin{bmatrix} \leftarrow \vec{c}^T \rightarrow \\ \leftarrow \vec{c}^T A \rightarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix}$

rank 2 if $T \neq 0$

$\Longrightarrow$ observable

The remaining question-parts below pertain to designing an observer for the system, i.e., finding $\vec{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ such that $\hat{\vec{x}}[t]$ in the observer system below develops into a good approximation of $\vec{x}[t]$.

The observer is

$$\underbrace{\begin{bmatrix} \hat{p}[t+1] \\ \hat{v}[t+1] \end{bmatrix}}_{\hat{\vec{x}}[t+1]} = A \underbrace{\begin{bmatrix} \hat{p}[t] \\ \hat{v}[t] \end{bmatrix}}_{\hat{\vec{x}}[t]} + \vec{b}u[t] + \underbrace{\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}}_{\vec{l}} (\hat{p}[t] - y[t]).$$

(turn to the next page for the next question-part.)

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4d) Suppose we set $l_1 = 0$. Is it possible to design a stable observer by setting l_2 to some appropriate value? (if so, provide and justify such a value; if not, explain why not). (4 points)

$\longrightarrow$ CLOSED LOOP MATRIX: $\hat{A} = A + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix} = \begin{bmatrix} 1+l_1 & T \\ l_2 & 1 \end{bmatrix}$

$\longrightarrow$ eigenvalues of $\hat{A}$: $(1+l_1-\lambda)(1-\lambda) - l_2 T = 0$

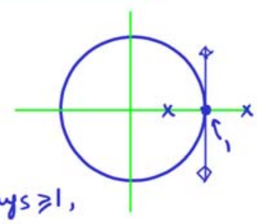
$$\Rightarrow (1-\lambda)^2 + l_1(1-\lambda) - l_2 T = 0$$

$$\Rightarrow (1-\lambda) = \frac{-l_1 \pm \sqrt{l_1^2 + 4 l_2 T}}{2}$$

$$\Rightarrow \lambda_{1,2} = 1 + \frac{l_1}{2} \pm \frac{\sqrt{l_1^2 + 4 l_2 T}}{2};$$

$\longrightarrow$ If $l_1 = 0$:

$$\lambda_{1,2} = 1 \pm \sqrt{l_2 T}:$$



$\rightarrow$ if $l_2 \geq 0$, then the xs are the e.v.s: one is always ≥ 1, hence unstable

$\longrightarrow$ if $l_2 < 0$, both evs are outside the unit circle, as shown $\Rightarrow$ unstable

$\longrightarrow$ Hence: NOT POSSIBLE TO DESIGN A STABLE OBSERVER WITH ANY CHOICE OF l_2 if $l_1 = 0$.

4e) Suppose we set $l_2 = 0$. Is it possible to design a stable observer by setting l_1 to some appropriate value? (if so, provide and justify such a value; if not, explain why not). (1 point)

$\longrightarrow \lambda_1, \lambda_2 = 1 + \frac{l_1}{2} \pm \frac{l_1}{2} = 1,\ 1 + l_1$

$\longrightarrow \lambda_1 = 1$ is always marginally stable $\Rightarrow$ BIBO unstable

$\longrightarrow$ NOT POSSIBLE

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4f) Suppose we set $l_1 = -2$.
(i) Is it possible to design a stable observer by setting l_2 to some appropriate value? Justify your answer.
(ii) Is it possible to design an **unstable** observer by setting l_2 to some appropriate value? Justify your answer. (2 points)

$$\lambda_{1,2} = \pm \frac{\sqrt{4+4l_2T}}{2} = \pm\sqrt{1+l_2T}$$

(i) STABLE: POSSIBLE: eg: $l_2 = -\frac{1}{2T} \Rightarrow \lambda_{1,2} = \pm\sqrt{1-1/2} = \pm\sqrt{\frac{1}{2}}$, both are <1 in magnitude.

(ii) UNSTABLE: POSSIBLE: eg $l_2 = \frac{1}{T} \Rightarrow \lambda_{1,2} = \pm\sqrt{2} = \pm 1.414$, both are >1 in magnitude.

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Problem 5 (10 points) PCA and SVD

5a) For each of the following, indicate if it is a valid **covariance matrix** or not. Justify your answers concretely. (2 points)

5a-i) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$: VALID INVALID

Explanation: $\sigma_1^2 < |\sigma_{12}|$

5a-ii) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$: VALID INVALID

Explanation: NOT SYMMETRIC

5a-iii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$: VALID INVALID

Explanation: $\sigma_1^2 < |\sigma_{12}|$

5a-iv) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$: VALID INVALID

Explanation:

→ EIGENVALUES : $(1-\lambda)^2 - 1 = 0 \Rightarrow 1-\lambda = \pm 1 \Rightarrow \lambda_{1,2} = 1 \pm 1 = 0, 2$

i.e, ≥ 0 .

→ ALTERNATIVE SOLN: $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ ← 0 mean cols (& rows)

$\longrightarrow S = A^T A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

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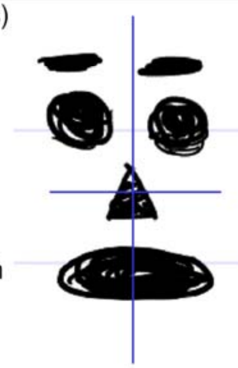
5b) PCA intuition: (2 points)

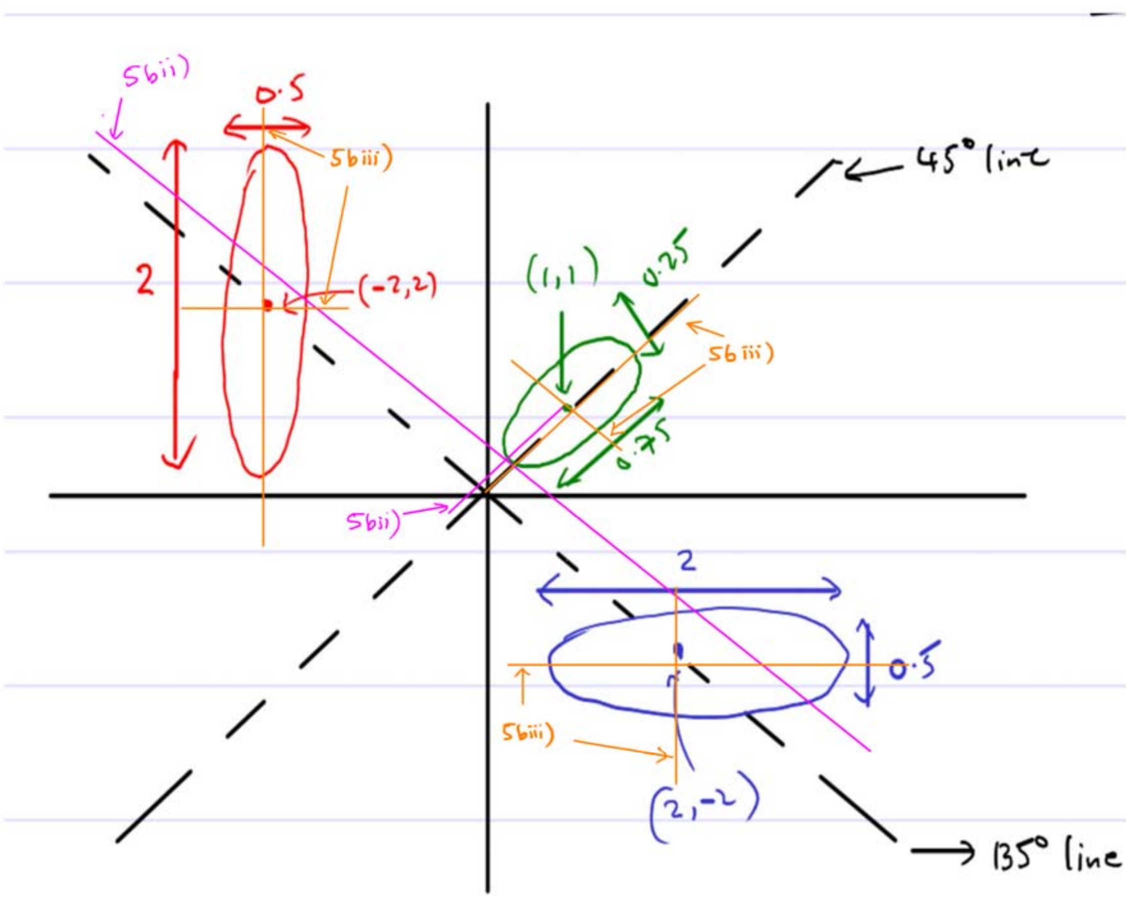
5b-i) Sketch two straight lines on the 2-d data on the right: one (longer) indicating the direction of the principal component, and another (shorter) indicating the direction of the 2nd principal component.



5b-ii) The data in the figure below are in three clusters, ie, in the ellipses with centroids/means of each cluster marked. Assume that each ellipse is uniformly populated by data points. The major and minor extents of the eliipses are also shown.

Suppose we run a PCA analysis on this data. Sketch lines on the figure indicating the direction of the principal component (longer) and the 2nd principal component (shorter). Label them 5b-ii).

5b-iii) Now suppose we first run k-means on the data (with k=3) and succeed in identifying the clusters correctly. We then run PCA separately on each cluster. Sketch lines **on each cluster** indicating the direction of the principal component (longer line) and the 2nd principal component (shorter) for each. Label them 5b-iii).



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5c) Compute the SVD of $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 1 & 1 \end{bmatrix}$ using the technique illustrated by example in class. Show your calculations clearly. Note: only the **full SVD**, correctly computed, will receive full credit. Leave numbers like $\sqrt{2}$ as $\sqrt{2}$, i.e., don't "simplify" them to 1.414! (6 points)

$$S = A^T A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}$$

$\rightarrow$ Eigenvalues of S: $(6-\lambda)^2 - 1 = 0 \Rightarrow 6 - \lambda = \pm 1 \Rightarrow \lambda_{1,2} = 6 \pm 1 = 7, 5$

$\rightarrow$ Eigenvectors of S: $(6-\lambda) p_1 + p_2 = 0 \Rightarrow p_2 = (\lambda - 6) p_1$

$\rightarrow$ for λ_1: $p_2 = p_1$

$\rightarrow$ for λ_2: $p_2 = -p_1$

$$\Rightarrow P = \underline{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} \times \frac{1}{\sqrt{2}}$$ to make cols. norm 1.

$\rightarrow$ $\underline{V = P}$, $\Sigma = \underline{\begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{5} \\ 0 & 0 \end{bmatrix}}$

$$A\vec{v}_1 = \sigma_1 \vec{u}_1 \Rightarrow \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \sqrt{7}\, \vec{u}_1 \Rightarrow \underline{\vec{u}_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}}$$

$$A\vec{v}_2 = \vec{\sigma}_2 \vec{u}_2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} = \sqrt{5}\, \vec{u}_2 \Rightarrow \underline{\vec{u}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}}$$

$\rightarrow$ $\vec{u}_3$: must be $\perp$ to $\vec{u}_1$ & $\vec{u}_2$. Say $\vec{u}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\rightarrow$ $\vec{u}_3 \perp \vec{u}_1 \Rightarrow 3a - b + 2c = 0 \Rightarrow 2c = b - 3a$.

$\rightarrow$ $\vec{u}_3 \perp \vec{u}_2 \Rightarrow -a - 3b = 0 \Rightarrow a = -3b$

$2c = 1 + 9 = 10$

$\Rightarrow c = 5$

$\rightarrow$ choose $b = 1 \Rightarrow a = -3$, $c = 5$ $\Rightarrow \underline{\vec{u}_3 = \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} \times \frac{1}{\sqrt{35}}}$

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(more space for 5c)

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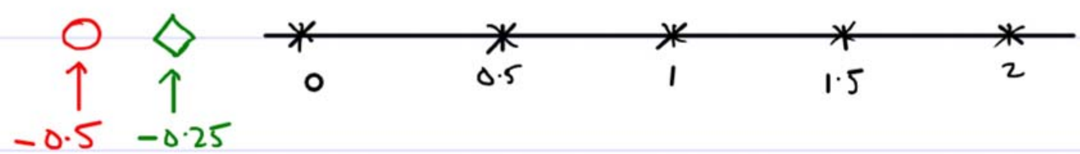
Problem 6 (5 points) k-means

The k-means algorithm has two steps:

- STEP 1: Assign each data point to the cluster/mean that is nearest; and
- STEP 2: Update each cluster's mean to be the average of the cluster's data. (If the cluster contains no data, then don't update the mean).

Run the k-means algorithm manually on the following data:



Each * represents a data point; the circle and diamond indicate the initial means of the k=2 clusters. Indicate the progress of k-means using the following template. ROUND 1, STEP 1 is filled out for you as an illustration. **Indicate clearly where the algorithm stops**. YOU SHOULD NOT FILL IN ANY STEPS AFTER THE ALGORITHM STOPS.

INIT
circle cluster: mean = -0.5, data = {empty}
diamond cluster: mean = -0.25, data = {empty}

ROUND 1, STEP 1

circle cluster: data = { **empty** }

diamond cluster: data = { **0, 0.5, 1, 1.5, 2** }

ROUND 1, STEP 2

circle cluster: mean = -0.5

diamond cluster: mean = 1

ROUND 2, STEP 1

circle cluster: data = { 0 }

diamond cluster: data = { 0.5, 1, 1.5, 2 }

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ROUND 2, STEP 2

circle cluster: mean = 0

diamond cluster: mean = 1.25

ROUND 3, STEP 1

circle cluster: data = { 0 , 0.5 }

diamond cluster: data = { 1, 1.5, 2 }

ROUND 3, STEP 2

circle cluster: mean = 0.25

diamond cluster: mean = 1.5

ROUND 4, STEP 1

circle cluster: data = { 0 , 0.5 }

diamond cluster: data = { 1 , 1.5, 2 }

→ NO CHANGE FROM RND 3 STEP 1 ⇒ FINISHED

ROUND 4, STEP 2

circle cluster: mean =

diamond cluster: mean =

ROUND 5, STEP 1

circle cluster: data = { }

diamond cluster: data = { }

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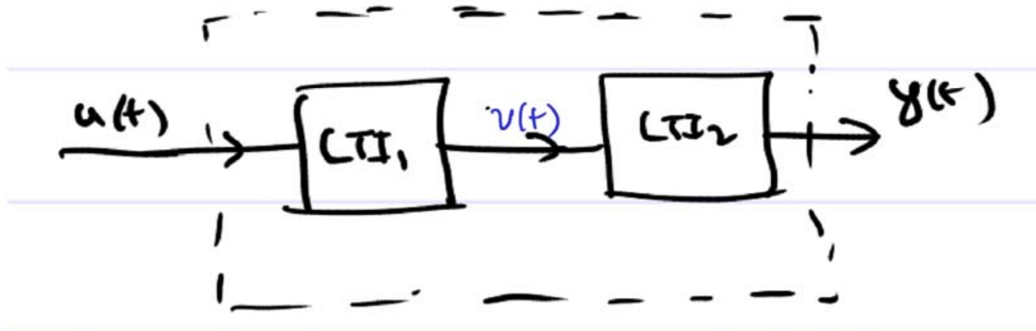
Problem 7 (15 points) – LTI systems

7a) Prove that the composition of two LTI systems is LTI. In other words, that if each block in the figure below is LTI, then $u(t) \mapsto y(t)$ is LTI. You must be clear and precise (and correct) in your reasoning for full credit. (3 points)



→ TI: $u(t-\tau) \xmapsto{LTI_1} v(t-\tau)$

$v(t-\tau) \xmapsto{LTI_2} y(t-\tau)$

$\Longrightarrow \underline{u(t-\tau) \xmapsto{LTI_1 \,\&\, LTI_2} y(t-\tau)}$ i.e., $u(t) \mapsto y(t)$ is T.I

→ LINEARITY:

→ SCALING:

$\alpha u(t) \xmapsto{LTI_1} \alpha v(t)$

$\alpha v(t) \xmapsto{LTI_2} \alpha y(t)$

$\Longrightarrow \underline{\alpha u(t) \xmapsto{LTI_1 \,\&\, LTI_2} \alpha y(t)}$ i.e., SCALING HOLDS FOR $u(t) \mapsto y(t)$

→ SUPERPOSITION: Say $u_1(t) \xmapsto{LTI_1} v_1(t) \xmapsto{LTI_2} y_1(t)$, and

$u_2(t) \xmapsto{LTI_1} v_2(t) \xmapsto{LTI_2} y_2(t)$

→ then: $(u_1(t) + u_2(t)) \xmapsto{LTI_1} (v_1(t) + v_2(t))$

$(v_1(t) + v_2(t)) \xmapsto{LTI_2} (y_1(t) + y_2(t))$

$\Longrightarrow \underline{(u_1(t) + u_2(t)) \xmapsto{LTI_1 + LTI_2} (y_1(t) + y_2(t))}$

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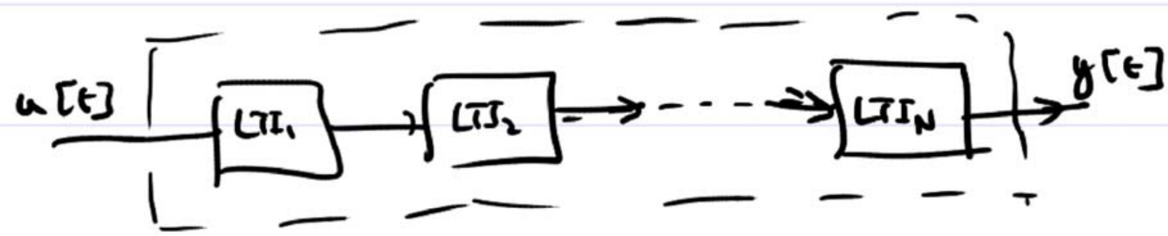
7b) Using reasoning similar to 7a, it is easy to show that the composition of N LTI systems, as depicted below, is LTI. (2 points)



Suppose all the internal LTI blocks LTI_1 to LTI_N are identical, with impulse response

$$h[t] = \begin{cases} 1, & t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the impulse response $h_c[t]$ of the composed system, i.e., from $u(t) \mapsto y(t)$. Show your reasoning clearly.

$\longrightarrow$ $h[t]$ simply delays the input by 1 $\Longrightarrow$ $u[t] \circledast h[t] = u[t-1]$

$\Longrightarrow \delta[t] \xmapsto[h[t]]{LTI_1} \delta[t-1] \xmapsto[h[t]]{LTI_2} \delta[t-2] \mapsto \cdots \xmapsto[h[t]]{LTI_N} \delta[t-N]$

$\Longrightarrow h_c[t] = \delta[t-N] \longleftarrow$

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7c) Given a discrete-time, causal, LTI system with impulse response h[t].

You are not told what h[t] is.

However, you **are** told that if the input u[t] to the system is chosen to be h[t], then the first five samples of the output y[t] are: y[0] = 1, y[1] = 0, y[2] = -2, y[3] = 0, y[4] = 3.

Find **two different** possible solutions for h[t] (only for t=0, …, 4) that satisfy the above condition. Are other solutions for t=0,…,4 also possible? Justify your answers and arguments clearly and precisely.

(4 points)

$$y[t] = h[t] \circledast h[t] = \sum_{i=0}^{t} h[i]\,h[t-i]$$

$$y[0] = h^2[0] = 1 \Rightarrow h[0] = \pm 1$$

$$y[1] = h[0]h[1] + h[1]h[0] = 2h[0]h[1] = 0 \Rightarrow h[1] = 0$$

$$y[2] = h[0]h[2] + h^2[1] + h[2]h[0] = 2h[0]h[2] = -2 \Rightarrow h[2] = -1/h[0] = \mp 1.$$

$$y[3] = h[0]h[3] + h[1]h[2] + h[2]h[1] + h[3]h[0] = 2h[0]h[3] = 0 \Rightarrow h[3] = 0$$

$$y[4] = 2h[0]h[4] + 2h[1]h[3] + h^2[2] = 3 \Rightarrow 2h[0]h[4] + 1 = 3 \Rightarrow h[4] = 1/h_0 = \pm 1$$

→ Hence two solutions are:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \;\&\; \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

→ NO OTHER SOLUTIONS ARE POSSIBLE (PER THE ABOVE DERIVATION).

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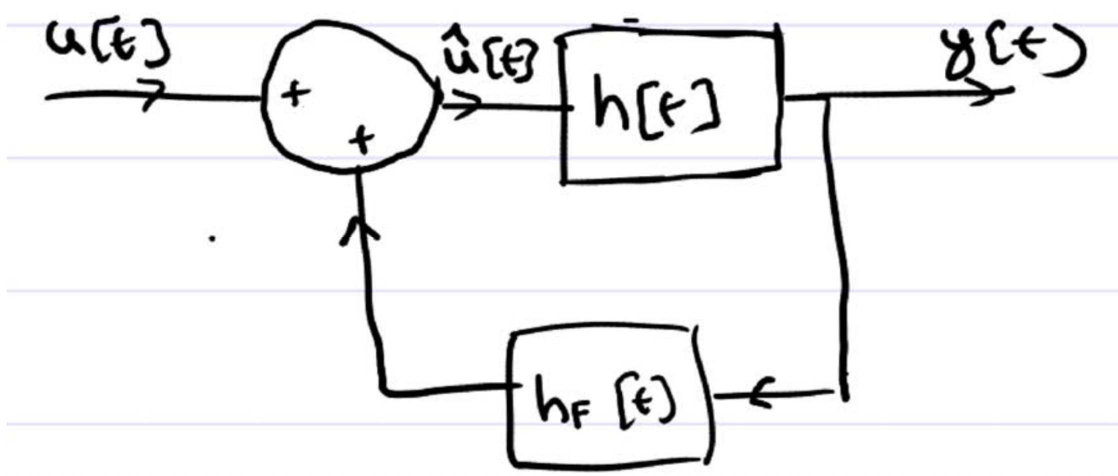
7d) Given two discrete-time LTI systems in a feedback loop (6 points)



with $h[t] = \begin{cases} 1, & t=0, \\ -1, & t=1, \\ 0, & \text{otherwise.} \end{cases}$, and $h_F[t] = \begin{cases} 1, & t=1, \\ 0, & \text{otherwise.} \end{cases}$

Please read through the expressions above for h[t] and $h_F[t]$ **very carefully** and make sure you understand them right. Note also that the feedback **adds** to the input.

7d-i) Assuming that $y[t]=0$ for all t<0, find the impulse response $h_c[t]$ of the closed-loop system (i.e., from $u(t) \mapsto y(t)$).

Hint: write out y[t] for t=0 ,…,15 at least (not required, but highly recommended), examine the values and use to devise a general formula or expression for y[t]. Be very careful to avoid mistakes in your calculations (please double and triple-check each step).

$\longrightarrow \hat{u}[t] = u[t] + y[t] \circledast h_F[t] = u[t] + y[t-1]$

$\longrightarrow y[t] = \hat{u}[t] \circledast h[t] = \hat{u}[t] - \hat{u}[t-1] = u[t] + y[t-1] - u[t-1] - y[t-2]$

$\longrightarrow \underline{y[t] = (y[t-1] - y[t-2]) + (u[t] - u[t-1])}$

$\longrightarrow$ Set $u[t] = \delta(t)$:

$\rightarrow y[0] = 1$; $y[1] = 1 - 1 = 0$; $y[2] = -1$; $y[3] = -1$;

$y[4] = 0$; $y[5] = 1$; $y[6] = 1$; $y[7] = 0$; $y[8] = -1$; $y[9] = -1$

$t \rightarrow$	0	1	2	3	4	5	6	7	8	9
$\rightarrow$ i.e., $y[0-8] =$	1	0	−1	−1	0	1	1	0	−1	−1

$\longrightarrow$ FROM HERE ON, $y[t] = y[t-1] - y[t-2]$

$\Longrightarrow$ for $t \geq 2$, $y[t]$ depends only on the previous 2 values $y[t-1]$ & $y[t-2]$

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$\longrightarrow$ SINCE $y[8] = y[2]$ & $y[9] = y[3]$, THE SEQUENCE MUST REPEAT, with $y[10] = y[4]$, $y[11] = y[5]$, AND SO ON :

$$\Longrightarrow h_c[t] = \begin{cases} 0 , & t < 0 \\ 1 , & t = 0 \\ 0 , & t = 1 \\ -1 , & t = 2 \\ -1 , & t = 3 \\ 0 , & t = 4 \\ 1 , & t = 5 \\ 1 , & t = 6 \\ 0 , & t = 7 \\ h_c[t-6], & t \geq 8 \end{cases}$$

7d-ii) Is the closed-loop system BIBO stable or BIBO unstable? If stable, explain why. If unstable, write out an input u[t] that will make y[t] → ∞ as t → ∞.

$\longrightarrow$ BIBO UNSTABLE :

$\longrightarrow$ JUST TRY $u[t] = h_c[t] \Longrightarrow y[t] = h_c[t] \circledast h_c[t] = \sum_{i=0}^{t} h_c[i]\, h_c[t-i]$

	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	← i
											1	0	−1	−1	0	1	1	0	−1	−1	0	1	1	← h[i]
	0	−1	−1	0	1	1	0	−1	−1	0	1													← h[−i]
0	−1	−1	0	1	1	0	−1	−1	0	1	1	0	−1	−1	0	1								← h[5−i]
						0	−1	−1	0	1	1	0	−1	−1	0	1	1	0	−1	−1	0	1		← h[11−i]

$\rightarrow$ NOTICE THAT $h[5-i] = h[i]$ for $i = 0, \cdots, 5$

$$\Longrightarrow y[5] = \sum_{i=0}^{5} h^2[i] = 4$$

$\longrightarrow$ BECAUSE $h[i]$ REPEATS WITH PERIOD 6, we HAVE $y[11] = 8$, $y[17] = 12$, $\cdots$

$y[6k-1] = 4k$, $k = 1, 2, \cdots$ $\Longrightarrow$ $y[t]$ BLOWS UP as $t \to \infty$ for BOUNDED INPUT $h[t]$.

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Problem 8 (20 points) DFT and Interpolation

Given N odd, $\vec{X} = F_N \vec{x}$, with $\vec{x}$ and $\vec{X}$ of size N.
F_N is the DFT matrix of size N, with the (k,l)[th] entry (numbering from 0) being ω_N^{-kl}, where $\omega_N = e^{j\frac{2\pi}{N}}$.

8a) Suppose X_i (capital X_i) are $X_i = \cos(\frac{2\pi i}{N}), \quad i = 0, \cdots, N-1.$

What is $\vec{x}$? (write an expression for the entries of $\vec{x}$).
Hint: if you apply a) the relation between the DFT matrix and its inverse, and b) the phasor-extracting properties of the DFT, this is a short exercise of a few lines. (4 points)

$$\vec{x} = F_N^{-1}\vec{X} = \frac{1}{N}F_N^{*}\vec{X} = \frac{1}{N}\overline{F_N}\,\vec{X}$$

$$\Longrightarrow \overline{\vec{x}} = \frac{1}{N}F_N\overline{\vec{X}} = \frac{1}{N}F_N\vec{X}$$

$\because \vec{X}$ as given above is real

$\longrightarrow F_N\vec{X}$ is just the (FORWARD) DFT of the entries in $\vec{X}$

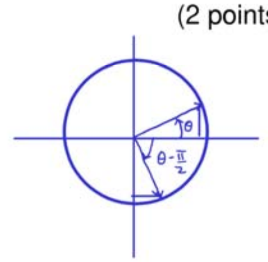
$\longrightarrow$ Using its phasor identifying properties, we get

$$\rightarrow \overline{\vec{x}} = \frac{1}{N}\begin{bmatrix}0\\ N/2\\ 0\\ \vdots\\ 0\\ N/2\end{bmatrix} = \begin{bmatrix}0\\ 1/2\\ 0\\ \vdots\\ 0\\ 1/2\end{bmatrix} \Longrightarrow \vec{x} = \overline{\vec{x}} = \begin{bmatrix}0\\ 1/2\\ 0\\ \vdots\\ 0\\ 1/2\end{bmatrix}$$

REAL

8b) Suppose $x_t = \sin(\dfrac{2\pi t}{N})$, $t = 0, \cdots, N-1$. What is $\vec{X}$? (2 points)

$\longrightarrow x_t$ are samples of $x(t) = \sin(\frac{2\pi}{N}t) = \cos(\frac{2\pi}{N}t - \frac{\pi}{2})$



$$\Longrightarrow \text{DFT coeffs are: } \vec{X} = \begin{bmatrix}0\\ \frac{N}{2}e^{-j\pi/2}\\ 0\\ \vdots\\ 0\\ \frac{N}{2}e^{+j\pi/2}\end{bmatrix}$$

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8c) We saw in class that if $\vec{x}$ are N=2M+1 samples of

$$x(t) = A_0 + \sum_{i=1}^{M} A_i \cos(2\pi i f_0 t + \theta_i), \tag{1}$$

i.e., if $x_i = x(i\Delta)$, with $\Delta = \dfrac{1}{N f_0}$ and i=0, …, N-1, then

$$X_i = \begin{cases} N A_0, & i = 0, \\ \frac{N}{2} A_i e^{\jmath\theta_i}, & i = 1, \cdots, M, \\ \bar{X}_{N-i}, & i = M+1, \cdots, N-1. \end{cases} \tag{2}$$

It is possible to express x(t) exactly in basis-function interpolation form as

$$x(t) = \sum_{i=0}^{N-1} x_i \, \phi_F(t - i\Delta). \tag{3}$$

Derive an expression for $\phi_F(t)$. Show every step of your derivation clearly.

Hint: suggested procedure:
(i) Express (1) using (2) – i.e., eliminate A_i in favour of X_i in (1).
(ii) Then, eliminate X_i in favour of x_i by applying the DFT relationship between $\vec{x}$ and $\vec{X}$.
(iii) Arrange the expression thus obtained for (1) to fit the form of (3) and identify an expression for $\phi_F(t)$. You will find the identity

$$\frac{1}{N}\left[1 + \sum_{i=1}^{M}\left(\omega_N^{ix} + \omega_N^{-ix}\right)\right] = \frac{\mathrm{sinc}(x)}{\mathrm{sinc}(\frac{x}{N})}$$

to be very useful. (No need to prove this identity here).

(14 points)

(i) $x(t) = A_0 + \sum_{i=1}^{M} A_i \cos(2\pi f_0 i t + \theta_i)$ ⟵ THIS IS (1)

$$= A_0 + \sum_{i=1}^{M}\left(\frac{A_i e^{j\theta_i}}{2} e^{j2\pi f_0 i t} + \frac{A_i}{2} e^{-j\theta_i} e^{-j2\pi f_0 i t}\right)$$

→ Using (2), we have: $A_0 = \dfrac{X_0}{N}$, $A_i e^{j\theta_i} = \dfrac{2X_i}{N}$, $i=1,\dots,M$

⟹ (1) BECOMES: $x(t) = \dfrac{1}{N}\left[X_0 + \sum_{i=1}^{M}\left(X_i e^{j2\pi f_0 i t} + \bar{X}_i e^{-j2\pi f_0 i t}\right)\right]$ ⟵ (4)

(ii) Since $\vec{X} = F_N \vec{x}$, we have: $X_i = \sum_{k=0}^{N-1} \omega_N^{-ik} x_k$ ⟵ (5)

and

(iii) → Put (5) into (4):

$\bar{x}_k = x_k \because$ real

$$x(t) = \frac{1}{N}\left[\underbrace{\sum_{k=0}^{N-1} x_k}_{X_0} + \sum_{i=1}^{M}\left[e^{j2\pi f_0 i t}\underbrace{\sum_{k=0}^{N-1}\omega_N^{-ik} x_k}_{X_i} + e^{-j2\pi f_0 i t}\underbrace{\sum_{k=0}^{N-1}\omega_N^{+ik} x_k}_{\bar{X}_i}\right]\right]$$

(space for 8c) $\left(\text{using } W_N = e^{j\frac{2\pi}{N}} \right)$

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$$\Rightarrow x(t) = \frac{1}{N}\left[\sum_{k=0}^{N-1} x_k + \sum_{i=1}^{M}\sum_{k=0}^{N-1}\left[e^{j2\pi f_0 i t}\, e^{-j\frac{2\pi}{N}ik} + e^{-j2\pi f_0 i t}\, e^{j\frac{2\pi}{N}ik}\right] x_k\right]$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} x_k + \sum_{k=0}^{N-1}\left(\sum_{i=1}^{M}\left[e^{j2\pi i(f_0 t - k/N)} + e^{-j2\pi i(f_0 t - k/N)}\right]\right) x_k\right]$$

$$= \sum_{k=0}^{N-1} \frac{1}{N}\left[1 + \sum_{i=1}^{M}\left[e^{j2\pi i(f_0 t - k/N)} + e^{-j2\pi i(f_0 t - k/N)}\right]\right] x_k$$

$$= \sum_{k=0}^{N-1} \frac{1}{N}\left[1 + \sum_{i=1}^{M}\left[e^{j\frac{2\pi}{N} i(Nf_0 t - k)} + e^{-j\frac{2\pi}{N} i(Nf_0 t - k)}\right]\right] x_k$$

$$= \sum_{k=0}^{N-1} \underbrace{\frac{1}{N}\left[1 + \sum_{i=1}^{M}\left[W_N^{i(Nf_0 t - k)} + W_N^{-i(Nf_0 t - k)}\right]\right]}_{\text{USING THE PROVIDED IDENTITY, WE GET}} x_k$$

$$x(t) = \sum_{k=0}^{N-1} \frac{\text{sinc}(Nf_0 t - k)}{\text{sinc}\left(\frac{Nf_0 t - k}{N}\right)} x_k = \sum_{k=0}^{N-1} \frac{\text{sinc}(t/\Delta - k)}{\text{sinc}\left(\frac{t/\Delta - k}{N}\right)} x_k \longleftarrow (6)$$

SAME

$\rightarrow$ DEFINE $\boxed{\phi_F(t) \triangleq \frac{\text{sinc}(t/\Delta)}{\text{sinc}(t/N\Delta)}}$; then $\phi_F(t-k\Delta) = \boxed{\frac{\text{sinc}(t/\Delta - k)}{\text{sinc}\left(\frac{t/\Delta - k}{N}\right)}}$

$\rightarrow$ HENCE (6) CAN BE WRITTEN AS

$$x(t) = \sum_{k=0}^{N-1} \phi_F(t - k\Delta)\, x_k \longleftarrow \text{which is (3)}$$

SID#________________________

(space for 8c) (this is the last page of the exam)