To do : Differential Equations with $u(t)$

Email : nareauphol.lin@

① Discretization

② Solving the general $u(t)$

Recall :

① $\frac{d}{dt}x(t) = \lambda x(t)$

② $\frac{d}{dt}x(t) = \lambda x(t) + u$

change of variables

③ $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$

Can we make ③ look ②? [or ①]

$x(t) = \boxed{e^{\lambda t}x_0 + be^{\lambda t}\int_0^t e^{-\lambda\tau}u_c(\tau)\,d\tau}$

Goal today : See why $x(t)$ for ③

Discretization

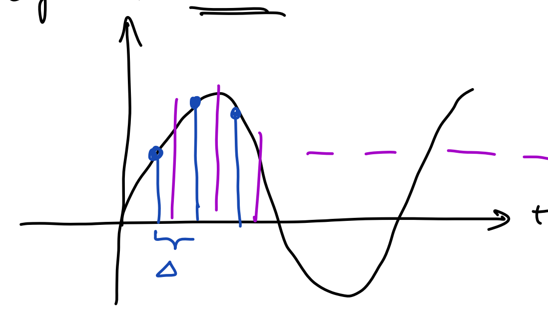
Method : Discretization

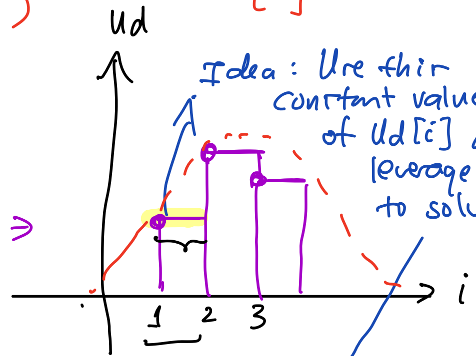
Continuous Time (CT) ⟺ Discrete Time (DT)

( ) [ ]

→ sampling

e.g. $u(t) = \sin(t)$





Idea : Use this constant value of $u_d[i]$ and leverage ② to solve

Problem : this is only for this "chunk" (timeframe)

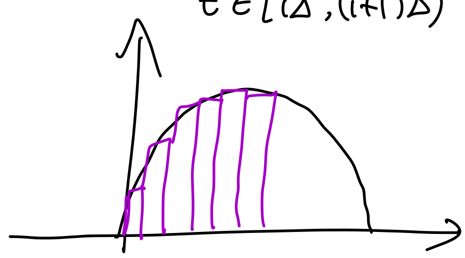
New problem : $\frac{d}{dt}x(t) = \lambda x(t) + b\,u_d[i]$

$t \in [i\Delta, (i+1)\Delta)$

$x(i\Delta) = x_d[i]$, $u(i\Delta) = u_d[i]$

$x(\Delta) = x_d[1]$

Hint : Change of Variables!



$$\frac{d}{dt}x(t) = \lambda\left(x(t) + \frac{bu_d[i]}{\lambda}\right)$$

let $\tilde{x}(t) = x(t) + \frac{bu_d[i]}{\lambda}$

$$\frac{d}{dt}x(t) = \lambda\tilde{x}(t) \qquad t \in [i\Delta, (i+1)\Delta)$$

$$\frac{d}{dt}\left[\tilde{x}(t) - \frac{bu_d[i]}{\lambda}\right] = \lambda\tilde{x}(t)$$

$$\frac{d}{dt}\tilde{x}(t) = \lambda\tilde{x}(t)$$

$$\tilde{x}(t) = \cancel{\tilde{x}(0)e^{\lambda t}} \Rightarrow \tilde{x}(i\Delta)e^{\lambda(t-i\Delta)}$$

(modified solution to ①
∵ we changed where our initial condition is)

[Same as last discussion but

$$\therefore \tilde{x}(i\Delta) = x(i\Delta) + \frac{bu_d[i]}{\lambda}$$

$$\therefore \tilde{x}(i\Delta) = x_d[i] + \frac{bu_d[i]}{\lambda}$$

$$\tilde{x}(t) = \left(x_d[i] + \frac{bu_d[i]}{\lambda}\right)e^{\lambda(t-i\Delta)}$$

$$(i+1)\Delta - i\Delta = \Delta$$

$$\therefore x(t) = \left(x_d[i] + \frac{bu_d[i]}{\lambda}\right)e^{\lambda(t-i\Delta)} - \frac{bu_d[i]}{\lambda}$$

$$\therefore x_d[i+1] = x((i+1)\Delta) =$$

$t = (i+1)\Delta$

$x(i\Delta) = x_d[i]$

$x((i+1)\Delta) = x_d[i+1]$

$$= \left(x_d[i] + \frac{bu_d[i]}{\lambda}\right)e^{\lambda\cancel{(t-i\Delta)}\,\Delta} - \frac{bu_d[i]}{\lambda}$$

$$= x_d[i]e^{\lambda\Delta} + \frac{bu_d[i]}{\lambda}e^{\lambda\Delta} - \frac{bu_d[i]}{\lambda}$$

$$= x_d[i]e^{\lambda\Delta} + \frac{bu_d[i]}{\lambda}\left(e^{\lambda\Delta} - 1\right)$$

(b) $\alpha = e^{\lambda\Delta}, \quad \beta = \frac{b(e^{\lambda\Delta}-1)}{\lambda}$

(c) $x_d[i+1] = \alpha x_d[i] + \beta u[i] \qquad x_d[3] =$

$$x_d[1] = \alpha x_d[0] + \beta u[0]$$

$$x_d[2] = \alpha x_d[1] + \beta u[1]$$

$$= \alpha^2 x_d[0] + \alpha\beta u[0] + \beta u[1]$$

$$= \alpha^2 x_d[0] + \beta(\alpha u[0] + u[1])$$

$$x_d[3] = \alpha x_d[2] + \beta u[2]$$
$$= \alpha^3 x_d[0] + \beta(\alpha^2 u[0] + \alpha u[1] + u[2])$$
$$\vdots$$
$$x_d[i] = \alpha^i x_d[0] + \beta \sum_{j=0}^{i-1} \alpha^{i-1-j} u[j]$$

(d) $\Delta = 2$, $t = 13$

t	2	4	6	8	10	12	14
i	1	2	3	4	5	6	7

$$\boxed{i = \left\lfloor \frac{t}{\Delta} \right\rfloor}$$ floor operation

$\lfloor 3.99 \rfloor = 3$

$\lfloor 3.14 \rfloor = 3$

$\lfloor 3.5 \rfloor = 3$

(e) $x(t) \approx x\left(\Delta \left\lfloor \frac{t}{\Delta} \right\rfloor\right) = x_d\left[\left\lfloor \frac{t}{\Delta} \right\rfloor\right]$

From (c) we know

$$x_d[i] = \alpha^i x_d[0] + \beta \sum_{j=0}^{i-1} \alpha^{i-1-j} u[j]$$

$$x(t) \approx x_d\left[\left\lfloor \frac{t}{\Delta} \right\rfloor\right] = \alpha^{\lfloor \frac{t}{\Delta} \rfloor} x_d[0] + \beta \sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} \alpha^{\lfloor \frac{t}{\Delta} \rfloor - 1 - j} u[j]$$

From (b) we know $\alpha = e^{\lambda\Delta}$, $\beta = \frac{b}{\lambda}(e^{\lambda\Delta} - 1)$

$$x(t) \approx x_d\left[\left\lfloor \frac{t}{\Delta} \right\rfloor\right] = (e^{\lambda\Delta})^{\lfloor \frac{t}{\Delta} \rfloor} x_d[0] + \frac{b}{\lambda}(e^{\lambda\Delta} - 1) \cdot \sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} \alpha^{\lfloor \frac{t}{\Delta} \rfloor - 1 - j} u[j]$$

$$\lim_{\Delta \to 0}$$