To do : System of Differential Equations   Email : nareauphol.lin @

1. Matrix-vector Form
2. Change of Variables (Change of Basis)

Recall :

1. $\frac{d}{dt}x(t) = \lambda x(t)$
2. $\frac{d}{dt}x(t) = \lambda x(t) + u$ [Dis 1B]
3. $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$ [Dis 2A + HW2]

$$\begin{cases} \frac{d}{dt}x_1(t) = 5x_1(t) \rightarrow x_1(t) = x_1(0)e^{5t} \\ \frac{d}{dt}x_2(t) = -5x_2(t) \rightarrow x_2(t) = x_2(0)e^{-5t} \end{cases}$$

$$\begin{cases} \frac{d}{dt}x_1(t) = 5x_1(t) - 5x_2(t) \\ \frac{d}{dt}x_2(t) = -5x_2(t) + 5x_1(t) \end{cases}$$

Diff. eqs are coupled!

Problem : If your diff eqs are coupled, you don't know how to solve.

$\vec{x} \in \mathbb{R}^2 \Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 5 & -5 \\ 5 & -5 \end{bmatrix}\vec{x}(t)$$

$$\vec{x}(t) \in (\mathbb{R} \rightarrow \mathbb{R})^n \Rightarrow \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \rightarrow \frac{d}{dt}\vec{x}(t) = \underline{A}\vec{x}(t)$$

n×n matrix

1(a) $\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\vec{x}(t)$

$$\begin{cases} \frac{d}{dt}x_1(t) = -9x_1(t) \Rightarrow x_1(t) = x_1(0)e^{-9t} \rightarrow -e^{-9t} \\ \frac{d}{dt}x_2(t) = -2x_2(t) \Rightarrow x_2(t) = x_2(0)e^{-2t} \rightarrow 3e^{-2t} \end{cases}$$

(b) $\frac{dy_1(t)}{dt} = -5y_1(t) + 2y_2(t)$

$\frac{dy_2(t)}{dt} = 6y_1(t) - 6y_2(t)$

$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}_{\text{"A"}}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

<u>Goal</u> : How do I convert $\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}$ into $\begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}$?

<u>Method</u> : Change of Variables (Change of Basis)

(c) $y_1(t) = -\tilde{y}_1(t) + 2\tilde{y}_2(t)$

$y_2(t) = 2\tilde{y}_1(t) + 3\tilde{y}_2(t)$

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}}_{\text{"V"}}\begin{bmatrix} \tilde{y}_1(t) \\ \tilde{y}_2(t) \end{bmatrix}$$

$\vec{y}(t) = V\vec{\tilde{y}}(t)$

$\vec{\tilde{y}}(t) = V^{-1}\vec{y}(t) = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}^{-1}\vec{y}(t)$

$$\downarrow$$

$$\begin{bmatrix} -3/7 & 2/7 \\ 2/7 & 1/7 \end{bmatrix}$$

(d) $\vec{\tilde{y}}(t) = V^{-1}\vec{y}(t)$

$\vec{\tilde{y}}(0) = V^{-1}\vec{y}(0)$

$= \begin{bmatrix} -3/7 & 2/7 \\ 2/7 & 1/7 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(e) $\frac{d}{dt}\vec{y}(t) = \underline{A}\,\vec{y}(t)$ from (b)

$$\left.\begin{array}{l} \vec{y}(t) = V\vec{\tilde{y}}(t) \\ \vec{\tilde{y}}(t) = V^{-1}\vec{y}(t) \end{array}\right\} \text{ from (c)}$$

Converted A to $V^{-1}AV$ via change of variables/basis

$$\frac{d}{dt}V\vec{\tilde{y}}(t) = AV\vec{\tilde{y}}(t)$$

$$V\frac{d}{dt}\vec{\tilde{y}}(t) = AV\vec{\tilde{y}}(t)$$

$$\frac{d}{dt}\vec{\tilde{y}}(t) = \underline{V^{-1}AV}\,\vec{\tilde{y}}(t)$$

$$\underbrace{\begin{bmatrix} -3/7 & 2/7 \\ 2/7 & 1/7 \end{bmatrix}}_{V^{-1}} \underbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}}_{V}$$

$$V^{-1}\begin{bmatrix} 9 & -4 \\ -18 & -6 \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}$$

eigenvalues of A!

$$\therefore \frac{d}{dt}\vec{\tilde{y}}(t) = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\vec{\tilde{y}}(t)$$

$$\frac{d}{dt}\tilde{y}_1(t) = -9\tilde{y}_1(t) \Rightarrow \tilde{y}_1(t) = \tilde{y}_1(0)e^{-9t} = -e^{-9t}$$

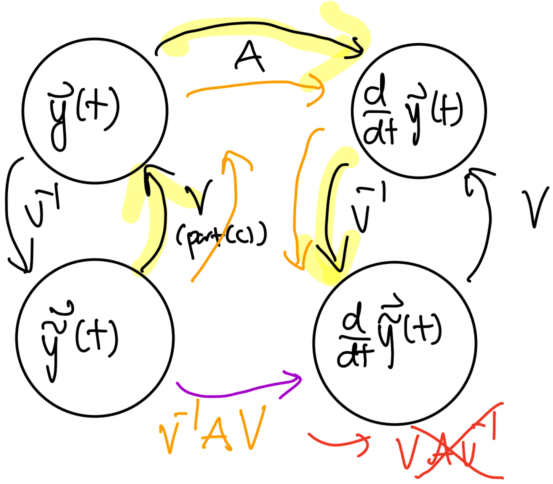
$$\frac{d}{dt}\tilde{y}_2(t) = -2\tilde{y}_2(t) \Rightarrow \tilde{y}_2(t) = \tilde{y}_2(0)e^{-2t} = 3e^{-2t}$$

(f) We have $\vec{\tilde{y}}(t)$, but we want $\vec{y}(t)$

$$\vec{y}(t) = V\vec{\tilde{y}}(t)$$

$$= \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix}$$

- We told you what V is! But what if no one told you? $\Rightarrow$ Next Monday



When multiplying matrices, you left multiply

- Diagonalization (Math 54)

$$A = V\Lambda V^{-1}$$

$$V^{-1}AV = \Lambda$$