To do : System of Differential Equations II   Email : nareauphol.lin @

① Review

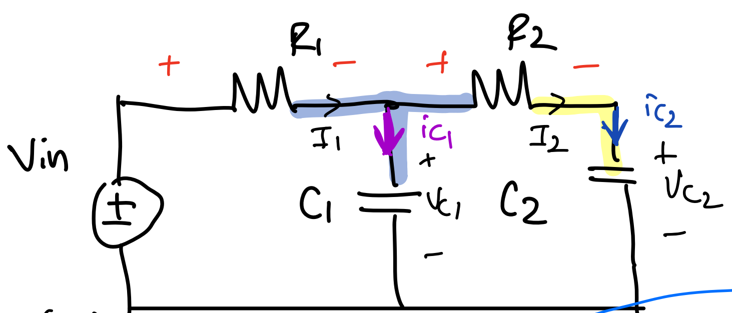
② Eigenbasis

$$\frac{d}{dt}x_1(t) = 5x_1(t) - 2x_2(t)$$
$$\frac{d}{dt}x_2(t) = 2x_1(t) - 3x_2(t)$$

coupled!

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \rightarrow \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 5 & -2 \\ 2 & -3 \end{bmatrix}}_{\text{“}A\text{”}}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

(a) $$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1R_1} - \frac{1}{C_1R_2} & \frac{1}{C_1R_2} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix}\begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix} + \begin{bmatrix} \frac{V_{in}}{C_1R_1} \\ 0 \end{bmatrix}$$



(C2)

KCL : $I_2 = i_{c_2}$

$$\frac{V_{R_2}}{R_2} = C_2\frac{dV_{c2}}{dt}$$

$$\frac{V_{c1} - V_{c2}}{R_2} = C_2\frac{dV_{c2}}{dt}$$

$$\frac{dV_{c2}}{dt} = \frac{V_{c1} - V_{c2}}{R_2C_2}$$ (1st equation)

(C1)

KCL : $I_1 = i_{c_1} + I_2$

$$\frac{V_{R_1}}{R_1} = C_1\frac{dV_{c1}}{dt} +$$

$$\frac{V_{in} - V_{c1}}{R_1} = C_1\frac{dV_{c1}}{dt} + \frac{V_{c1} - V_{c2}}{R_2}$$

$$\frac{V_{in} - V_{c1}}{R_1} - \frac{V_{c1} - V_{c2}}{R_2} = C_1\frac{dV_{c1}}{dt}$$ (2nd eqs)

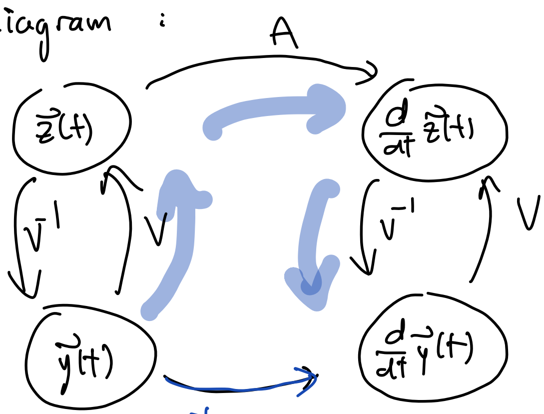
$$\frac{dV_{c1}}{dt} = \frac{V_{in} - V_{c1}}{C_1R_1} - \left(\frac{V_{c1} - V_{c2}}{C_1R_2}\right)$$

$$= \frac{V_{in}}{C_1R_1}$$

(b) $\vec{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad \vec{z}(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

(c) Recall diagram :



$V^{-1}AV$ = Diagonal Matrix

V matrix consists of eigenvectors of A $\Rightarrow$ $V^{-1}AV$ will be always be diagonal

$\rightarrow$ diagonal of eigenvalues of A : $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

(c) $A = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$$\det\begin{bmatrix} -5-\lambda & 2 \\ 6 & -6-\lambda \end{bmatrix} = 0$$

$(-5-\lambda)(-6-\lambda) - 12 = 0$

$\lambda^2 + 11\lambda + 18 = 0$

$(\lambda+9)(\lambda+2) = 0$

$\lambda_1 = -9, \ \lambda_2 = -2$

plug back in & solve nullspace

$\lambda_1 = -9$

$$\begin{bmatrix} -5-(-9) & 2 \\ 6 & -6-(-9) \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}$$

$e.\vec{v}_1 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\lambda_2 = -2, \ e\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

☆ Sum of eigenvalues = Sum of diagonal of A [Trace]

$$\therefore\ V = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}, \quad \text{is } V = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \text{ valid? } \underline{\text{Yes}}$$

$$D = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} -2 & 0 \\ 0 & -9 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} -3/7 & 2/7 \\ 2/7 & 1/7 \end{bmatrix}$$

(d), (e)

$$\begin{tikzcd}
\vec{z}(t) \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{z}(t) \arrow[d, bend right, "V^{-1}"'] \\
\vec{y}(t) \arrow[u, bend right, "V"'] \arrow[r] & \frac{d}{dt}\vec{y}(t) \arrow[u, bend right, "V"']
\end{tikzcd}$$

From (b), $\vec{z}(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

$$V^{-1}AV = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\left.\begin{aligned} A\vec{v}_1 &= \lambda_1 \vec{v}_1 \\ A\vec{v}_2 &= \lambda_2 \vec{v}_2 \\ &\ \vdots \\ A\vec{v}_n &= \lambda_n \vec{v}_n \end{aligned}\right\}$$

Last time:

$$\frac{d}{dt}\vec{z}(t) = A\vec{z}(t)$$

$$\frac{d}{dt}V\vec{y}(t) = AV\vec{y}(t)$$

$$\frac{d}{dt}\vec{y}(t) = \underbrace{V^{-1}AV}_{D}\vec{y}(t)$$

Today

$$A\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$$

$$\frac{d}{dt}\vec{z}(t) = A\vec{z}(t)$$

$$\downarrow$$

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\vec{y}(t)$$

$$y_1(t) = y_1(0)e^{-9t}$$

$$y_2(t) = y_2(0)e^{-2t}$$

$$\vec{y}(0) = V^{-1}\vec{z}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\therefore\ \left.\begin{aligned} y_1(t) &= -e^{-9t} \\ y_2(t) &= 3e^{-2t} \end{aligned}\right\}.$$

(f) $\vec{y}(t) = \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix}$

$$\begin{aligned} \vec{z}(t) &= V\vec{y}(t) \\ &= \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix} \end{aligned}$$

$$\begin{tikzcd} \vec{z}(t) \arrow[r, bend left, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{z}(t) \arrow[d, bend right, "V^{-1}"'] \\ \vec{y}(t) \arrow[u, bend right, "V"'] \arrow[r] & \frac{d}{dt}\vec{y}(t) \arrow[u, bend right, "V"'] \end{tikzcd}$$

$V^{-1}AV = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$V = [\vec{v}_1, \ldots \vec{v}_n]$ where $\vec{v}_i$ are eigenvectors of A

Summary

① Write diff eqn in matrix-vector form $\left(\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)\right)$

→ Circuits

② Find eigenvalues & eigenvectors of A

③ Construct V using eigenvectors, + calculate V^{-1}

④ Convert your problem into the eigen-world [basis]

→ $V^{-1}AV = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

→ Don't forget to convert initial conditions

→ Solve it

⑤ Convert your solution back into the original world [basis]

$$\frac{d}{dt}\vec{z}(t) = A\vec{z}(t) + b$$

$$b = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

$$\frac{d}{dt} V\vec{y}(t) = AV\vec{y}(t) + b$$

$$\frac{d}{dt}\vec{y}(t) = V^{-1}\left(AV\vec{y}(t) + b\right)$$

$$= V^{-1}AV\vec{y}(t) + V^{-1}b$$

$$= D\vec{y}(t) + \underline{V^{-1}b} \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\frac{d}{dt}y_1(t) = \lambda_1 y_1(t) + 2 \longrightarrow \frac{d}{dt}x(t) = \lambda x(t) + u$$

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t)$$