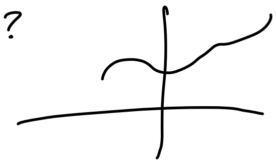
To do : Module 2 !

1. Discretization

2. Generalizing discretization

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t)$$

$v(t) = ?$



Intro

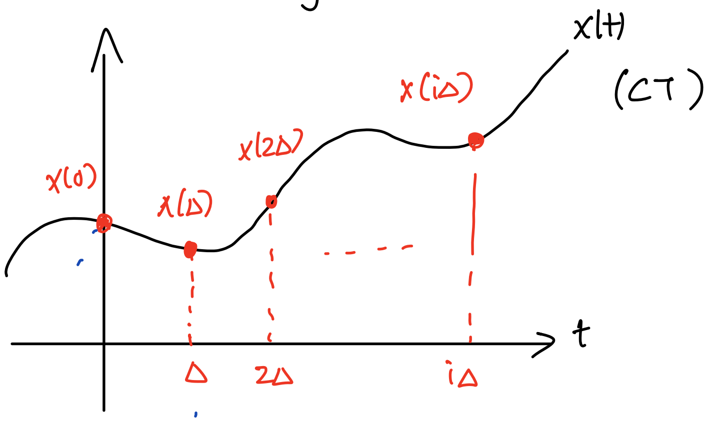
Discretization via Sampling

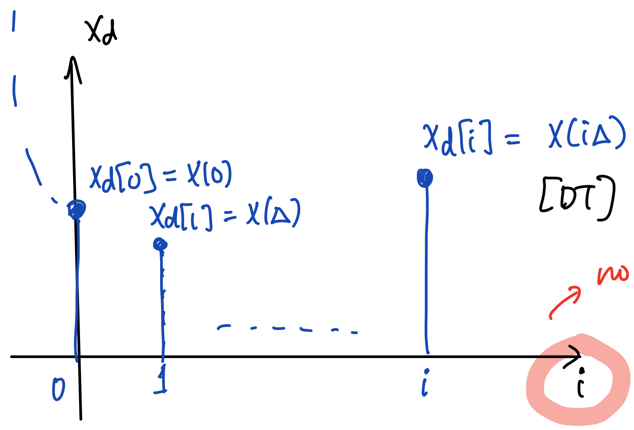
Analog → Digital

→ Sensor readings (Continuous Time) CT

→ Computer logic [Discrete Time] DT

Method : Sampling (Disc 2A)





no time, constant values at a given index depending the sample value.

Recall : $\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$

$$x(t) = e^{\lambda(t-t_0)}x(t_0) + b\int_{t_0}^{t} u(\theta)\, e^{\lambda(t-\theta)}\, d\theta$$, t_0 is starting time

$t \in [i\Delta, (i+1)\Delta)$, what is $X_d[i+1]$ given we know $X_d[i] = X(i\Delta)$

$\therefore$ $u(t)$ is a piecewise constant, $u(t) = u(i\Delta) = u_d[i]$ over $t \in [i\Delta, (i+1)\Delta)$

$$\therefore \quad \frac{d}{dt}x(t) = \lambda x(t) + b u_d[i]$$

(a)
$$x(t) = e^{\lambda(t-i\Delta)} x(i\Delta) + b \int_{i\Delta}^{t} \overset{u_d[i]}{\cancel{u(\theta)}} e^{\lambda(t-\theta)} d\theta$$

$$x(t) = e^{\lambda(t-i\Delta)} x_d[i] + b u_d[i] \int_{i\Delta}^{t} e^{\lambda(t-\theta)} d\theta$$

Goal: $x_d[i+1] = ?? \; x_d[i]$

$$x((i+1)\Delta) = e^{\lambda\Delta} x_d[i] + b u_d[i] \int_{i\Delta}^{(i+1)\Delta} e^{\lambda((i+1)\Delta - \theta)} d\theta$$

$$x_d[i+1] = e^{\lambda\Delta} x_d[i] + b u_d[i] \left(\frac{e^{\lambda\Delta} - e^{0}}{\lambda} \right)$$

$$\boxed{x_d[i+1] = e^{\lambda\Delta} x_d[i] + b u_d[i] \left(\frac{e^{\lambda\Delta} - 1}{\lambda} \right)}$$

$$\frac{d}{dt}\underbrace{x(t)}_{CT} = \lambda \underbrace{x(t)}_{CT} + b u_d[i], \quad t \in [i\Delta, (i+1)\Delta)$$

$\big\downarrow$ discretization

$$x_d[i+1] = e^{\lambda\Delta} x_d[i] + b u_d[i] \left(\frac{e^{\lambda\Delta} - 1}{\lambda} \right)$$

Discretizing scalar diff. eq from $t \in [i\Delta, (i+1)\Delta)$

(b)
$$\frac{d}{dt}\vec{x}(t) = \underbrace{A_c \vec{x}(t) + \vec{b}_c}_{\text{continuous}} u_d[i]$$

$$\big\downarrow$$

$$\vec{x}_d[i+1] = A_d \vec{x}_d[i] + \tilde{b}_d u_d[i]$$

Goal: Find A_d and $\vec{b}_d$ given A_c and $\vec{b}_c$ in CT

<u>Hint</u>: Eigenbasis

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$\left(\, 1 \quad 2 \,\right)$

$$\frac{d}{dt}\vec{x}(t) = A_c\vec{x}(t) + \vec{b}_c u_d[i] \quad t\in[i\Delta,(i+1)\Delta)$$

$$\frac{d}{dt}V\tilde{\vec{x}}(t) = A_c V\tilde{\vec{x}}(t) + \vec{b}_c u_d[i]$$

$$\begin{tikzcd} \vec{x}(t) \arrow[r, "A_c"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x}(t) \arrow[d, bend right, "V^{-1}"'] \\ \tilde{\vec{x}}(t) \arrow[u, bend right, "V"'] \arrow[r, "D=V^{-1}AV"'] & \frac{d}{dt}\tilde{\vec{x}}(t) \arrow[u, bend right, "V"'] \end{tikzcd}$$

$$\frac{d}{dt}\tilde{\vec{x}}(t) = V^{-1}A_cV\tilde{\vec{x}}(t) + \underline{V^{-1}\vec{b}_c}\, u_d[i]$$

$$= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \tilde{\vec{x}}(t) + \tilde{\vec{b}}_c u_d[i]$$

Look at kth row $\rightarrow$

$$\frac{d}{dt}\tilde{x}_k(t) = \lambda_k\tilde{x}_k(t) + \left(\tilde{\vec{b}}_c u_d[i]\right)_k \quad \leftarrow \text{index } k \text{ of the vector}$$

$$\boxed{\begin{aligned} &\frac{d}{dt}\underbrace{x(t)}_{CT} = \lambda \underbrace{x(t)}_{CT} + b u_d[i], \quad t\in[i\Delta,(i+1)\Delta) \\ &\quad\Big\downarrow \text{ discretization} \\ &x_d[i+1] = e^{\lambda\Delta}x_d[i] + b u_d[i]\left(\frac{e^{\lambda\Delta}-1}{\lambda}\right) \end{aligned}}$$

$$\tilde{x}_{d_k}[i+1] = e^{\lambda_k\Delta}\tilde{x}_{d_k}[i] + \left(\tilde{\vec{b}}_c u_d[i]\right)_k\left(\frac{e^{\lambda_k\Delta}-1}{\lambda_k}\right)$$

$$\vdots \qquad\qquad \vdots$$

$$\tilde{x}_{d_n}[i+1] = e^{\lambda_n\Delta}\tilde{x}_{d_n}[i] + \left(\quad '' \quad\right)_n\left(\frac{e^{\lambda_n\Delta}-1}{\lambda_n}\right)$$

Stack the discretized row as a vector

$$\tilde{\vec{x}}_d[i+1] = \begin{bmatrix} e^{\lambda_1\Delta} & & & 0 \\ & \ddots & & \\ & & e^{\lambda_k\Delta} & \\ & & & \ddots \\ 0 & & & e^{\lambda_n\Delta} \end{bmatrix}\tilde{\vec{x}}_d[i] + \begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & & & 0 \\ & \ddots & & \\ & & \frac{e^{\lambda_k\Delta}-1}{\lambda_k} & \\ & & & \ddots \\ 0 & & & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix}\tilde{\vec{b}}_c u_d[i]$$

$$\underbrace{V\tilde{\vec{x}}_d[i+1]}_{\vec{x}_d[i+1]} = V\begin{bmatrix} \quad \downarrow \quad \end{bmatrix}\tilde{\vec{x}}_d[i] + V\begin{bmatrix} \quad \downarrow \quad \end{bmatrix}\tilde{\vec{b}}_c u_d[i]$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\vec{x}_d[i+1] = \underbrace{V\left[\quad\right]\vec{V}^{-1}}_{A_d}\vec{x}_d[i] + \underline{V\left[\qquad\right]}V^{-1}\vec{b}_c u_d[i]$$

Recall

$$\frac{d}{dt}\vec{x}(t) = \underset{\text{continuous}}{A_c}\vec{x}(t) + \vec{b}_c u_d[i]$$

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \tilde{b}_d u_d[i]$$

$$\begin{tikzcd}
\vec{x}(t) \arrow[r, "A_c"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x}(t) \arrow[d, bend right, "V^{-1}"'] \\
\tilde{x}(t) \arrow[u, bend right, "V"'] \arrow[r] & \frac{d}{dt}\tilde{x}(t) \arrow[u, bend right, "V"']
\end{tikzcd}$$

$$D = V^{-1}AV$$

$$A_d = V\begin{bmatrix} e^{\lambda_1\Delta} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n\Delta}\end{bmatrix}V^{-1}$$

$$\vec{b}_d = V\begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{e^{\lambda_n\Delta}-1}{\lambda_n}\end{bmatrix}V^{-1}\vec{b}_c$$

- V is the matrix that contains the eigenvectors of A_c
- λ_i are the eigenvalues of A_c

1(c) $\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i]$

$$A_d^2 u_d[0] + A_d u_d[1] + u_d[2] + \dots$$

$$\vec{x}_d[1] = A_d x_d[0] + \vec{b}_d u_d[0]$$

$$\vec{x}_d[2] = A_d(A_d x[0] + \vec{b}_d u_d[0]) + \vec{b}_d u[1]$$

$$= A_d^2 x[0] + A_d\vec{b}_d u_d[0] + \vec{b}_d u[1]$$

$$= A_d^2 x[0] + (A_d u_d[0] + u[1])\vec{b}_d$$

$$\tilde{x}_d[i] = A_d^i x[0] + \left(\sum_{j=0}^{i-1} u[j]A_d^{i-1-j}\right)\tilde{b}_d \rightarrow \text{dis2A question}$$

$$\vec{x}_d[3] = A_d^3 x[0] + (A_d^2 u_d[0] + A_d u[1] + \overset{A_d^0}{u[2]})b_d$$

$$\sum_{j=0}^{i-1} u_d[j]A_d^{i-1-j}$$