To do: Stability and System ID

① System ID

② Stability

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$$

Goal: Learn what A and $\vec{b}$ are

$$\xrightarrow[u[i]]{\text{inputs}} \boxed{A,\ \vec{b}} \xrightarrow[\vec{x}[i+1]]{\text{observation}}$$

↳ include some noise

1(a) $x[i+1] = a\,x[i] + b\,u[i]$, given we know $x[0]$

$$\left.\begin{aligned} x[1] &= a\,x[0] + b\,u[0] \\ x[2] &= a\,x[1] + b\,u[1] \\ &\ \vdots \\ x[\ell] &= a\,x[\ell-1] + b\,u[\ell-1] \end{aligned}\right\} \quad \underbrace{\begin{bmatrix} x[1] \\ \vdots \\ x[\ell] \end{bmatrix}}_{\text{observations}}^{\vec{s}} = \underbrace{\begin{bmatrix} x[0] & u[0] \\ \vdots & \vdots \\ x[\ell-1] & u[\ell-1] \end{bmatrix}}_{\text{Data Matrix } D} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{p}\ \text{(parameter)}} + \begin{bmatrix} w[0] \\ \vdots \\ w[\ell-1] \end{bmatrix}$$

$\therefore\ \vec{s} = D\vec{p}$

$\therefore\ \vec{p} = D^{-1}\vec{s}$ ✓

$$\begin{aligned} x[i+1] &= a\,x[i] + b\,u[i] + w[i] \\ x[1] &= a\,x[0] + b\,u[0] + w[0] \\ x[2] &= a\,x[1] + b\,u[1] + w[1] \\ &\ \vdots \\ x[\ell] &= a\,x[\ell-1] + b\,u[\ell-1] + \underbrace{w[\ell-1]}_{\text{noise}} \end{aligned}$$

$\vec{s} = D\vec{p} + \cancel{\vec{w}}$ → Least Squares

$\vec{p} = (D^TD)^{-1}D^T\vec{s}$

↑ realistic

(b) $x[i+1] = a x[i] + b_1 u_1[i] + b_2 u_2[i]$

$$x[1] = a x[0] + b_1 u_1[0] + b_2 u_2[0]$$

$$\vdots$$

$$x[l] = a x[l-1] + b_1 u_1[l-1] + b_2 u_2[l-1]$$

$$\begin{bmatrix} x[1] \\ \vdots \\ x[l] \end{bmatrix} = \begin{bmatrix} x[0] & u_1[0] & u_2[0] \\ \vdots & \vdots & \vdots \\ x[l-1] & u_1[l-1] & u_2[l-1] \end{bmatrix} \begin{bmatrix} a \\ b_1 \\ b_2 \end{bmatrix}$$

$$\vec{s} = D \vec{\varphi}$$

$$\therefore \vec{p} = (D^T D)^{-1} D^T \vec{s}$$

problem

→ no guarantee that $D^T D$ is invertible

→ e.g. if $\vec{u_1}$ and $\vec{u_2}$ are linearly dependent on each other → D not full rank

→ show $\text{rank}(D) = \text{rank}(D^T D)$

(d) $\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i]$

$$x_1[i+1] = a_{11} x_1[i] + a_{12} x_2[i] + b_1 u[i]$$

$$x_2[i+1] = a_{21} x_1[i] + a_{22} x_2[i] + b_2 u[i]$$

$$\begin{cases} x_1[1] = a_{11} x_1[0] + a_{12} x_2[0] + b_1 u[0] \\ x_2[1] = a_{21} x_1[0] + a_{22} x_2[0] + b_2 u[0] \end{cases}$$

$$\vdots$$

$$x_1[l] = a_{11} x_1[l-1] + a_{12} x_2[l-1] + b_1 u[l-1]$$

$$x_2[l] = a_{21} x_1[l-1] + a_{22} x_2[l-1] + b_2 u[l-1]$$

$$\begin{bmatrix} x_1[1] \\ x_2[1] \\ \vdots \\ x_1[\ell] \\ x_2[\ell] \end{bmatrix} = \begin{bmatrix} x_1[0] & x_2[0] & 0 & 0 & u[0] & 0 \\ 0 & 0 & x_1[0] & x_2[0] & 0 & u[0] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1[\ell-1] & x_2[\ell-1] & 0 & 0 & u[\ell-1] & 0 \\ 0 & 0 & x_1[\ell-1] & x_2[\ell-1] & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ b_1 \\ b_2 \end{bmatrix}$$

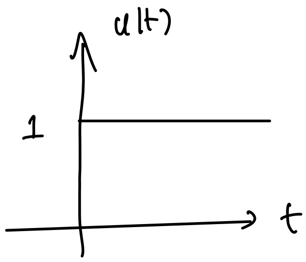
$\uparrow$ $u[\ell-1]$

OR

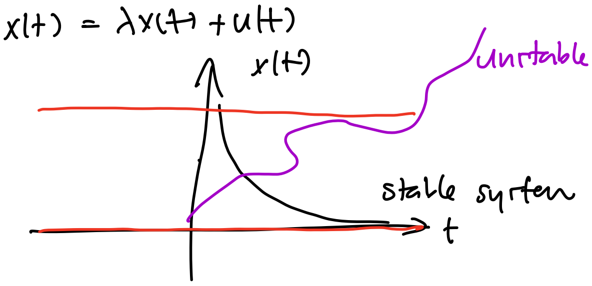
$$\begin{bmatrix} x_1[1] & x_2[1] \\ \vdots & \vdots \\ x_1[\ell] & x_2[\ell] \end{bmatrix} = \begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ \vdots & \vdots & \vdots \\ x_1[\ell-1] & x_2[\ell-1] & u[\ell-1] \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ b_1 & b_2 \end{bmatrix}$$

<u>Stability</u> → CT & DT

Ask yourself : For any given bounded input ($u(t)$, $u[i]$), will my output blow up?



$$\frac{d}{dt}x(t) = \lambda x(t) + u(t)$$



bounded input, bounded output

<u>BIBO Stability</u> : the solution to the CT system $x_c(t)$ is BIBO stable if $|x_c(t)| < \infty$ (CT)

the state $x_d[i]$ is BIBO stable if $|x_d[i]| < \infty$ [DT]

2(a) $$\frac{dV_c(t)}{dt} = -2V_c(t) + 2u(t) \rightarrow \frac{d}{dt}x(t) = \lambda x(t) + u(t)$$

$$V_c(t) = V_c(0)e^{\lambda t} + \int_0^t e^{-2(t-\theta)} 2u(\theta)\, d\theta$$

Goal: Show that $|V_c(t)| < \infty$ Fa19 MT2

$$V_c(t) = V_c(0)\, e^{\overset{-2}{\cancel{\lambda}}t} + 2\int_0^t e^{-2(t-\theta)} u(\theta)\, d\theta$$

$$|V_c(t)| = \left| V_c(0)\, e^{\overset{-2}{\cancel{\lambda}}t} + 2\int_0^t e^{-2(t-\theta)} u(\theta)\, d\theta \right|$$

$$\leq \underbrace{\left|V_c(0)\, e^{\overset{-2}{\cancel{\lambda}}t}\right|}_{\text{bounded because of } e^{-2t}} + 2\left|\int_0^t e^{-2(t-\theta)} u(\theta)\, d\theta\right|$$

$|u(t)| \leq k < \infty$

$$\leq \left|V_c(0)\, e^{\lambda t}\right| + 2\int_0^t e^{-2(t-\theta)} |u(\theta)|\, d\theta$$

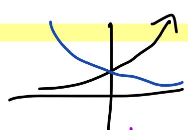
$$\leq \left|V_c(0)\, e^{\lambda t}\right| + 2k\int_0^t e^{-2(t-\theta)}\, d\theta$$

$$\leq \underbrace{\left|V_c(0)\, e^{\lambda t}\right|}_{\text{bounded output}} + \underbrace{2k\left(1 - e^{-2t}\right)}_{\text{bounded output}} < \infty$$

Continuous Time	Discrete time
$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$	$\vec{x}[i+1] = A\vec{x}[i] + bu[i]$
→ CT system is stable if ALL Re(eigenvalues of A) < 0	→ DT system is stable if ALL $\lvert\lambda_i\rvert$ are < 1
	λ_i ← eigenvalues of A

$\frac{d}{dt}x(t) = \lambda x(t)$

$x(t) = x_0 e^{\lambda t}$



$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$

eigenvalues of A

$$\vec{x}(t) = \begin{bmatrix} 5e^{-2t} + 9e^{-9t} \\ 6e^{-2t} + 7e^{-9t} \end{bmatrix} \rightarrow \text{if } x_1(t) = 5e^{-2t} + 9e^{9t}$$

$x[0] = 0$

$u[0] = 1,\ u[i] = 0 \quad \forall i \geq 1$

$x[i+1] = 2x[i] + u[i]$

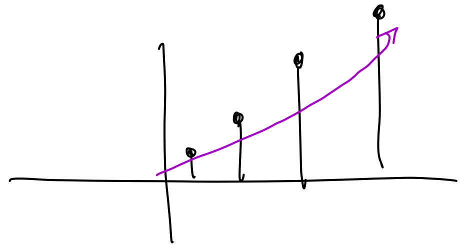
$x[1] = 2x[0] + u[0] = 1$

$x[2] = 2x[1] + \cancel{u[1]}^{0} = 2$

$x[3] = 2x[2] = 4$

$\vdots$

$x[\ell] = 2^{\ell-1}$



0.5

$x[i+1] = \cancel{2}\, x[i] + u[i]$

$x[0] = 0$

$u[0] = 1,\ u[i] = 0$
$\forall i \geq 1$

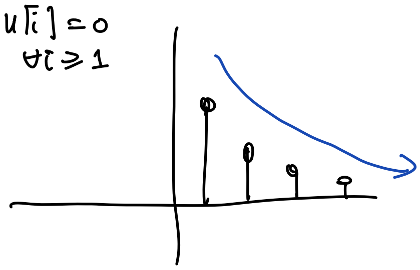
$x[1] = 0.5 x[0] + u[0] = 1$

$x[2] = 0.5 x[1] + \cancel{u[1]} = 0.5$

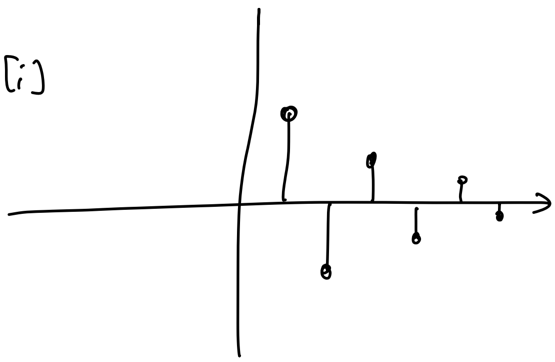
$x[3] = 0.5 x[2] = 0.5 \cdot 0.5 = 0.25$

$\vdots$

$x[l] = 0.5^{l-1}$

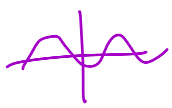


$x[i+1] = -0.5 x[i] + u[i]$



$\frac{d}{dt} x(t) = \lambda x(t)$

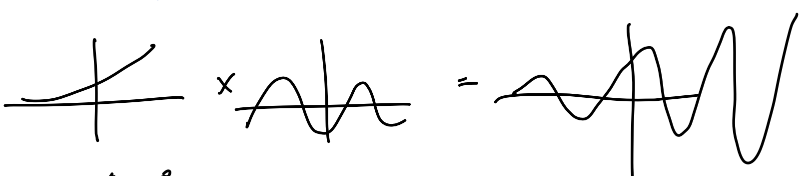
$\lambda = 1+j$



$x(t) = \cancel{x_0} e^{(1+j)t}$

$= e^{t} \cdot \boxed{e^{jt}} \rightarrow e^{j\omega t}$

$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta$



$\lambda = -1+j$

$x(t) = e^{-t} e^{jt}$