To do

① Recap Stability

② Stability via Feedback

☆ Midterm : Next Monday , CSM Review Session 7 PM Tuesday (tomorrow) @ HP auditorium.

Stability Recap

- BIBO stability : the solution to the CT system $x(t)$ is stable if $|x(t)| < \infty$

real part of eigenvalues of A is "less than 0 DT system $x[i]$ " " " $|x[i]| < \infty$

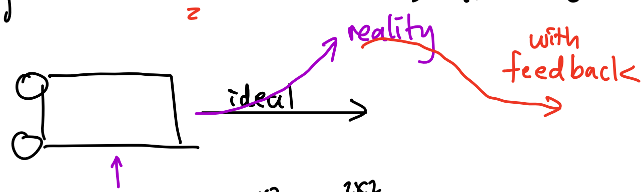
- CT : $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$

$\rightarrow$ $Re(\lambda_i) < 0$ for all eigenvalues λ_i of A

- DT : $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$

$\rightarrow$ $|\lambda_i| < 1$ for all eigenvalues λ_i of A

e.g. $x[i+1] = \underset{2}{3}x[i] + u[i]$ $\rightarrow$ not stable



e.g. $\frac{d}{dt}\vec{x}(t) = \overset{2\times2}{A}\vec{x}(t) + \overset{2\times2}{B}\vec{u}(t)$, eigenvalues of A is $\underline{2}$ and -2

① Pick your input $\vec{u}(t) = \underline{K}\vec{x}(t)$ $\rightarrow$ $\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + BK\vec{x}(t)$$
$$= \underbrace{(A + BK)}_{\text{your new "A"}}\vec{x}(t)$$

② Solve for the new eigenvalues of your new system $A+BK$

$\rightarrow$ characteristic polynomial will have $k_1 .. k_4$ term inside

e.g. $\underbrace{\lambda^2 - k_1k_2\lambda + k_3k_4}$ $+ 2 = 0$

③ Pick my k values to get the eigenvalues I want

<u>Goal:</u> $\lambda_1 = -2,\ \lambda_2 = -3 \Rightarrow (\lambda+2)(\lambda+3) = 0$

$\therefore\ \underline{\lambda^2 + 5\lambda + 6} = 0$

new system is stable!

↑ you can pick any goal

$\therefore\ k_1 k_2 = -5,\ k_3 k_4 + 2 = 6$

1(a) $x[i+1] = \boxed{0.9}\,x[i] + \cancel{u[i]}^{\,0} + w[i]$ $\quad |\lambda| < 1 \rightarrow$ stable.

$u[i] = 0,\ x[0] = 0,\ |w[i]| \le \varepsilon$

$x[1] = 0.9\,x[0] + w[0] = w[0]$

$x[2] = 0.9\,w[0] + w[1]$

$x[3] = 0.9^2 w[0] + 0.9\,w[1] + \overset{0.9^0}{w[2]}$

$\vdots$

$$x[i] = \sum_{k=0}^{i-1} 0.9^k\, w[i-1-k]$$

$|x[i]| < \infty$

$$\therefore |x[i]| = \left| \sum_{k=0}^{i-1} 0.9^k w[i-1-k] \right|$$

$$\le \sum_{k=0}^{i-1} |0.9^k w[i-1-k]|$$

$$\le \sum_{k=0}^{i-1} 0.9^k |w[i-1-k]| = \sum_{k=0}^{i-1} 0.9^k \varepsilon$$

$$= \varepsilon \sum_{k=0}^{i-1} 0.9^k = \varepsilon\left(\frac{1}{1-0.9}\right) = 10\varepsilon < \infty$$

$a_1 \xrightarrow{\times r} a_2 \xrightarrow{\times r} a_3$ $\quad \boxed{\dfrac{a_1}{1-r}}$

(b) $u[i] = f\,x[i]$, $x[i+1] = 0.9\,x[i] + u[i] + w[i]$

$\therefore\ x[i+1] = 0.9\,x[i] + f\,x[i] + w[i]$

$= \underbrace{(0.9+f)}_{\lambda}\, x[i] + w[i]$

(c) $x[i+1] = \underline{(0.9+f)\,x[i]} + w[i]$

$|x[i]|$ to be minimum, what is f?

minimise $(0.9+f)\,x[i]$, pick $f = -0.9 \rightarrow x[i+1] = w[i]$

$x[i] = w[i-1] \le \varepsilon$

(d) $x[i+1] = \overset{\text{scalar}}{(3+f)}\,x[i] + w[i]$

Without feedback $\rightarrow$ Unstable

With feedback $\rightarrow$ Unstable / Stable depending on f

(e) $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i]$

$\vec{u}[i] = F\vec{x}[i]$

$$\vec{x}[i+1] = A\vec{x}[i] + BF\vec{x}[i] + \vec{w}[i]$$

$$= \underbrace{(A+BF)}_{A_{CL}}\vec{x}[i] + \vec{w}[i]$$

$$\therefore\ A + BF = A_{CL}$$

$$\therefore\ F = B^{-1}(A_{CL} - A)$$

$\hookrightarrow$ B must be invertible

2 $\vec{x}[i+1] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u[i]$

(a) $\vec{x}[0] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\qquad \vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$\vec{x}[1] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ u[0] \end{bmatrix}$

$\ell = 4,\ u[0] = 1,\ u[1] = 2,\ u[2] = 3,\ u[3] = 4$

$\vec{x}[2] = \begin{bmatrix} 0 \\ 0 \\ u[0] \\ u[1] \end{bmatrix}$

$\vec{x}[3] = \begin{bmatrix} 0 \\ u[0] \\ u[1] \\ u[2] \end{bmatrix}$

$\vec{x}[4] = \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix}$

(b) $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$\vec{x}[1] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ u[0] \end{bmatrix}$

$u[0] = 4$

$\ell = 1$

(c) $\vec{x}[0] = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$\ell = 4,\ u[0] = 1$

$u[1] = 2$

$u[2] = 3$

$u[3] = 4$

$\vec{x}[1] = \begin{bmatrix} 2 \\ 1 \\ 0 \\ u[0] \end{bmatrix}$

$\vec{x}[2] = \begin{bmatrix} 1 \\ 0 \\ u[0] \\ u[1] \end{bmatrix}$

(d) $\vec{x}[\ell] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, $\ell = 4$ with $u[0] = a$, $u[1] = b$, $u[2] = c$, $u[3] = d$

$\vec{x}[3] = \begin{bmatrix} 0 \\ u[0] \\ u[1] \\ u[2] \end{bmatrix}$

$\vec{x}[4] = \begin{bmatrix} u[0] \\ \vdots \\ u[3] \end{bmatrix}$

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \vec{u}[i]$$

$$\vec{x}[1] = \underline{A\vec{x}[0]} + \underline{b}\,\underline{u[0]} \qquad \vec{x}[0] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ u[0] \end{bmatrix}$$

$$\vec{x}[2] = A\vec{x}[1] + \vec{b}u[1]$$

$$= \begin{bmatrix} 0 \\ 0 \\ u[0] \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ u[1] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u[0] \\ u[1] \end{bmatrix}$$

$\vec{x}[3]$:

$\vec{x}[\ell] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$!