To do :

Past Exams : Fa19, Sp21, Fa21

1. Feedback
2. Controllability

1(a) $\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + \vec{w}[i]$

Stability: All $|\lambda_i| < 1$

Recall eigenvalues can be found via $\det(A - \lambda I) = 0$

$\det\begin{pmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{pmatrix} = 0$

$\lambda^2 + \lambda - 2 = 0 \Rightarrow \boxed{\lambda_1 = -2}, \lambda_2 = 1$ $\therefore$ not stable

(b) $u[i] = [f_1 \ \ f_2]\,\vec{x}[i]$

$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \underbrace{\underset{2\times 1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underset{1\times 2}{[f_1 \ \ f_2]}} \vec{x}[i] + \vec{w}[i]$

$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix} \vec{x}[i] + \vec{w}[i]$

$= \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{CL}} \vec{x}[i] + \vec{w}[i]$

(c) $\boxed{\lambda_1 = \frac{1}{2}, \ \lambda_2 = -\frac{1}{2}}$ $\longrightarrow$ $(\lambda + \frac{1}{2})(\lambda - \frac{1}{2}) = 0$

$\lambda^2 - \frac{1}{4} = 0$ [desired]

$\therefore \det(A_{CL} - \lambda I) = 0$

$\det \begin{bmatrix} f_1 - \lambda & 1+f_2 \\ 2 & -1-\lambda \end{bmatrix} = 0$

$(f_1 - \lambda)(-1-\lambda) - 2(1+f_2) = 0$

$\lambda^2 + (1-f_1)\lambda + (-f_1 - 2f_2 - 2) = 0$

$\therefore 1 - f_1 = 0 \rightarrow f_1 = 1$

$-f_1 - 2f_2 - 2 = -\frac{1}{4} \rightarrow f_2 = \frac{-11}{8}$

(d) Yes because with feedback, your eigenvalues are now $-\frac{1}{2}, \frac{1}{2}$

$|\lambda_i| < 1$ is satisfied

(e) $\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]$

→ can't Stabilize a system → show that eigenvalue(s) not affected by feedback

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} [f_1 \ f_2]}_{\begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix}} \vec{x}[i]$$

$$\vec{x}[i+1] = \begin{bmatrix} f_1 & 1+f_2 \\ 2+f_1 & -1+f_2 \end{bmatrix} \vec{x}[i]$$

$$\det(A_{CL} - \lambda I) = 0$$

$$\det \begin{bmatrix} f_1 - \lambda & 1+f_2 \\ 2+f_1 & -1+f_2-\lambda \end{bmatrix} = 0$$

$$(f_1-\lambda)(-1+f_2-\lambda) - (1+f_2)(2+f_1) = 0$$

$$f_1f_2 - f_1 - f_1\lambda - \lambda f_2 + \lambda + \lambda^2 - f_1f_2 - f_1 - 2f_2 - 2 = 0$$

$$\lambda^2 + (1-f_1-f_2)\lambda - 2(1+f_1+f_2) = 0 \Rightarrow \lambda^2 + \lambda - 2 = 0$$

looks familiar

↓ Math leap of intuition

$$(\lambda+2)(\lambda-(1+f_1+f_2)) = 0$$

one of the λ has always be -2 regardless of f_1, f_2

→ ∴ not possible to stablize system

Controllability : Can I get to anywhere I want at a certain time?

span!

$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$, $\vec{x}[0] = \vec{0}$

$\vec{x}[1] = \vec{b}u[0]$ — the basis spans $\mathbb{R}^2$

$\vec{x}[2] = A\vec{b}u[0] + \vec{b}u[1]$

$\vec{x}[3] = A^2\vec{b}u[0] + A\vec{b}u[1] + \vec{b}u[2]$ — $A: 2\times 2 \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\vec{x}[i] = \begin{bmatrix} x \\ y \end{bmatrix}$

⋮

$$\vec{x}[i] = A^{i-1}\tilde{b}\,u[0] + A^{i-2}\tilde{b}\,u[1] + \dots \tilde{b}\,u[i-1]$$

you control

Controllability Matrix ($\mathcal{C}$) $= [A^{i-1}\tilde{b},\ A^{i-2}\tilde{b},\ \dots\ \tilde{b}]$

→ if controllable, the controllability matrix must span the space entirely

→ the columns are linearly independent to each other

→ if controllable, the matrix is full rank

2 (a) $\vec{x}[i+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i]$, $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\mathcal{C} = [A^2 b,\ Ab,\ b]$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$ → not full rank → not controllable.

$$\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

(c) $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some ℓ NO → you cannot ever reach -2 given $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix}$, $\vec{x}[2] = \begin{bmatrix} 4 \\ -6+2u[0] \\ -3+2u[1] \end{bmatrix}$, $\vec{x}[3] = \begin{bmatrix} 8 \\ \dots \\ \dots \end{bmatrix}$

(b) $\vec{x}[i] = \begin{bmatrix} 2^i \\ -3x_1[i-1] + 1 \cdot x_3[i-1] \\ 1 \cdot x_2[i-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2u[i-1] \end{bmatrix}$

$\vec{x}[i] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= \begin{bmatrix} 2^i \\ -3x_i[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix}$$

span $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(d) $\vec{x}[2] = \begin{bmatrix} 4 \\ -6+2u[0] \\ -3+2u[1] \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -6+2u[0] \\ -3+2u[1] \end{bmatrix}$ We can control

can be any value

$$= \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \text{span}\left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$$