To do: Upper Triangularization ☆ Warning: Extremely Complicated discussion

Recall: We solved $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$ using a change of basis V

where $V = [\vec{v}_1 \dots \vec{v}_n]$ contains the eigenvectors of A

$\therefore V^{-1}AV = D = \begin{bmatrix}\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$ where $\lambda_1 \dots \lambda_n$ are the eigenvalues of A

→ Implied Assumption: A is diagonalizable

If A is NOT diagonalizable, then we don't have enough vectors for V

→ e.g. $\begin{bmatrix}1 & 1 \\ 0 & 1\end{bmatrix}$

$\therefore$ Must use UT to solve the system of diff. eqs

Why?

$$\frac{d}{dt}\tilde{x}(t) = \begin{bmatrix}\lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}\tilde{x}(t)$$

$$\frac{d}{dt}\tilde{x}_n(t) = \lambda_n\tilde{x}_n(t) \longrightarrow \tilde{x}_n(t) = \tilde{x}_n(0)e^{\lambda_n t}$$

simplifies to

$$\frac{d}{dt}\tilde{x}_{n-1}(t) = \lambda_{n-1}\tilde{x}_{n-1}(t) + c\tilde{x}_n(t) \Rightarrow \frac{d}{dt}x(t) = \lambda x(t) + u(t) \checkmark$$

Goal: Find a new basis U such that $U^{-1}AU = \begin{bmatrix}\lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$

Step 1: Find an eigenvalue - eigenvector pair of U

$\begin{bmatrix}1 \\ -1 \\ 0\end{bmatrix}$ in (a) is an eigenvector of $U \Rightarrow \lambda = 0$

Step 2: Construct an orthonormal basis U using the eigenvector found in step 1 (G.S.)

→ Run G.S. on the eigenvector + basis vectors of $\mathbb{R}^3$ (in this example)

$$\begin{bmatrix}1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{G.S.} \begin{bmatrix}1/\sqrt{2} & & & \\ -1/\sqrt{2} & -, & \cancel{\vec{0}}, & - \\ 0 & & & \end{bmatrix} = U$$

(always check orthonormal)

discard this vector

using given example : $U = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b)/(d) : Compute $U^T M U$ → notice that ∵ U is orthonormal

$\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{U^{-1} M U}$ $\qquad$ ∴ $U^T = U^{-1}$

$$U = \left[\begin{array}{c|cc} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{array}\right] = [\vec{u}_1, R]$$

$\vec{u}_1$ ↳ eigenvector of M $\qquad$ R

∴ $R = \begin{bmatrix} \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 0 \\ 0 & 1 \end{bmatrix}$

∴ $M\vec{u}_1 = \lambda_1 \vec{u}_1$

$$U^T M U = \begin{bmatrix} \vec{u}_1^T \\ R^T \end{bmatrix} M [\vec{u}_1, R]$$

$$= \begin{bmatrix} \vec{u}_1^T \\ R^T \end{bmatrix} [M\vec{u}_1 \quad MR]$$

$$= \begin{bmatrix} \vec{u}_1^T M \vec{u}_1 & \vec{u}_1^T MR \\ R^T M \vec{u}_1 & R^T MR \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u}_1^T \lambda_1 \vec{u}_1 & \vec{u}_1^T MR \\ R^T \lambda_1 \vec{u}_1 & R^T MR \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \underbrace{\cancel{\vec{u}_1^T \vec{u}_1}}_{1} & \vec{u}_1^T MR \\ \lambda_1 \underbrace{\cancel{R^T \vec{u}_1}}_{\vec{0}} & R^T MR \end{bmatrix}$$

$R = [\vec{u}_2, \vec{u}_3]$

$R^T = \begin{bmatrix} \vec{u}_2^T \\ \vec{u}_3^T \end{bmatrix}$ ∴ $R^T \vec{u}_1 = \begin{bmatrix} \vec{u}_2^T u_1 \\ \vec{u}_3^T u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$U^T M U = \begin{bmatrix} \lambda_1 & \vec{u}_1^T MR \\ \vec{0} & R^T MR \end{bmatrix}$$

(d) $U^T M U = \begin{bmatrix} \lambda_1 & \vec{u}_1^T MR \\ \vec{0} & R^T MR \end{bmatrix}$

since U is orthonormal

∴ $UU^T = I = U^T U$

$$M = U \begin{bmatrix} \lambda_1 & \vec{u}_1^T MR \\ 0 & R^T MR \end{bmatrix} U^T$$

$$= U \begin{bmatrix} \lambda_1 & \boxed{\vec{a}^T} \\ 0 & Q \end{bmatrix} U^T$$

∴ $Q = R^T MR$

$\boxed{\vec{a}^T = \vec{u}_1^T MR}$ doesn't matter

(b) $U^T M U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5/6 & \sqrt{2}/6 \\ 0 & \sqrt{2}/6 & 2/3 \end{bmatrix}$

eigenvalue of M

$\vec{0}$ $\quad Q$

Notice that Q is not guaranteed to be upper triangular

$\rightarrow$ ∴ we need to run step ① and ② again to make Q upper triangular

(c)

$$\therefore \quad Q = \begin{bmatrix} 5/6 & \sqrt{2}/6 \\ \sqrt{2}/6 & 2/3 \end{bmatrix}$$

Step 1 : Find eigenvalue / eigenvector pair of Q

$(\lambda_Q, \vec{v_Q}) \rightarrow 2\times1$

Step 2 : Run G.S using $\vec{v_Q}$

$$\therefore \left[\vec{v_Q}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] \xrightarrow{\text{G.S.}} \left[\vec{v_Q}, \underset{\downarrow\, Y}{-}, \cancel{\vec{0}}\right] = U_Q$$

reduce dimension by 1

$$\therefore U_Q = \left[\vec{v_Q} \mid Y\right]$$

$$\therefore U_Q^T Q U_Q = \begin{bmatrix} \lambda_Q & - \\ 0 & - \end{bmatrix} \longrightarrow \therefore Q = U_Q \begin{bmatrix} \lambda_Q & - \\ \vec{0} & - \end{bmatrix} U_Q^T$$

∴ (c) I know from the first time we ran the algo. that

$$M = U \begin{bmatrix} \lambda_1 & \vec{a}^T \\ \vec{0} & Q \end{bmatrix} U^T$$

∴ Plug Q into the matrix

$$\therefore M = U \begin{bmatrix} \lambda_1 & \vec{a}^T \\ \vec{0} & U_Q \begin{bmatrix} \lambda_Q & - \\ \vec{0} & - \end{bmatrix} U_Q^T \end{bmatrix} U^T$$

$$= U \left(\begin{bmatrix} 1 & 0 \\ 0 & U_Q \end{bmatrix} \begin{bmatrix} \lambda_1 & \vec{a}^T U_Q \\ \vec{0} & \begin{bmatrix} \lambda_Q & - \\ 0 & - \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U_Q^T \end{bmatrix} \right) U^T$$

$$= \left(U \begin{bmatrix} 1 & 0 \\ 0 & U_Q \end{bmatrix} \right) \begin{bmatrix} \lambda_1 & - & - \\ 0 & \lambda_Q & - \\ 0 & 0 & - \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & U_Q^T \end{bmatrix} U^T \right)$$

Recall U from earlier that $U = [\vec{u_1}, R]$

$$\therefore \quad M = \left([\vec{u_1}, R] \begin{bmatrix} 1 & 0 \\ 0 & U_Q \end{bmatrix} \right) \begin{bmatrix} \lambda_1 & - & - \\ 0 & \lambda_Q & - \\ 0 & 0 & - \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & U_Q^T \end{bmatrix} \begin{bmatrix} \vec{u_1}^T \\ R^T \end{bmatrix} \right)$$

$$= \begin{bmatrix} \vec{u_1}, & RU_Q \end{bmatrix} \begin{bmatrix} \lambda_1 & - & - \\ 0 & \lambda_Q & - \\ 0 & 0 & - \end{bmatrix} \begin{bmatrix} \vec{u_1}^T \\ U_Q^T R^T \end{bmatrix}$$

Again recall that $U_Q = [\vec{v_Q}, Y]$ as we defined earlier

$$\therefore \quad M = \begin{bmatrix} \vec{u_1}, & R[\vec{v_Q}, Y] \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \vec{u_1}^T \\ \begin{bmatrix} \vec{v_Q}^T \\ Y^T \end{bmatrix} R^T \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \vec{u_1}, & R\vec{v_Q}, & RY \end{bmatrix}}_{\text{new basis "U"}} \begin{bmatrix} \lambda_1 & - & - \\ 0 & \lambda_Q & - \\ 0 & 0 & - \end{bmatrix} \underbrace{\begin{bmatrix} \vec{u_1}^T \\ \vec{v_Q}^T R^T \\ Y^T R^T \end{bmatrix}}_{\text{but transposed}}$$

$$= U_{final} \begin{bmatrix} \lambda_1 & - & - \\ 0 & \lambda_Q & - \\ 0 & 0 & - \end{bmatrix} U_{final}^T$$

orthonormal

$$U_{final} = [\vec{u_1}, R\vec{v_Q}, RY]$$

eigenvector of M

rest of the G.S ran on $\vec{u_1}$

eigenvector of the smaller sub matrix Q

rest of the G-S ran on $\vec{v_Q}$

$$= [\underbrace{\vec{u_1}}_{3\times1\ \checkmark}, \underbrace{\underbrace{R}_{3\times2}\underbrace{\vec{v_Q}}_{2\times1}}_{3\times1\ \checkmark}, \underbrace{R}_{3\times2}\underbrace{Y}_{2\times1}] \Rightarrow 3\times3$$

3×1

$$\begin{matrix} 4\times4 \\ \text{matrix} \end{matrix} = [\vec{u_1}, R\vec{v_Q}, RY\underbrace{\vec{v_R}}_{\text{eigenvector of smaller submatrix } R \text{ got from } Q}, RYZ]$$

rest of G.S. ran on $\vec{v_R}$

Summary :

① Find eigenvalue - eigenvector pair of M (or A)

② Put the eigenvector with basis vector and run G.S.

discard the $\vec{0}$

$$\rightarrow \underbrace{\left[\vec{u}_1, \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\\vdots\\0\end{bmatrix} \cdots\right]}_{n+1 \text{ vectors}} \xrightarrow{\text{G.S.}} \left[\vec{u}_1, -, -, \cdots, \cancel{\vec{0}}, -, -\right]$$

③ Repeat ① and ② on smaller submatrix until the 2x2 case

→ Don't forget to reduce reduce dimension in step ②

Using example worksheet : $Q = \begin{bmatrix} \sqrt{3}/6 & \sqrt{6}/3 \\ -\sqrt{6}/3 & \sqrt{3}/3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/3 & \sqrt{6}/3 \\ -\sqrt{6}/3 & \sqrt{3}/3 \end{bmatrix}^T$

$$\therefore \quad M = \underbrace{\begin{bmatrix} \sqrt{2}/2 & -\sqrt{6}/6 & \sqrt{3}/3 \\ -\sqrt{2}/2 & -\sqrt{6}/6 & \sqrt{3}/3 \\ 0 & \sqrt{6}/3 & \sqrt{3}/3 \end{bmatrix}}_{U \text{ final numerically}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{6}/6 & \sqrt{3}/3 \\ -\sqrt{2}/2 & -\sqrt{6}/6 & \sqrt{3}/3 \\ 0 & \sqrt{6}/3 & \sqrt{3}/3 \end{bmatrix}^T$$