<u>To do</u> : Orthonormality Properties

<u>Recall</u> : For a nxn orthonormal square matrix U, the following properties hold :

$$\left[\begin{array}{ll} ① & U^T U = U U^T = I_{n\times n} \\ ② & U^T = U^{-1} \end{array}\right.$$

1(a) $\vec{y_1} = U\vec{x_1}$

$\vec{y_2} = U\vec{x_2}$

$$\langle \vec{y_1}, \vec{y_2} \rangle = \vec{y_1}^T \vec{y_2} = \vec{y_2}^T \vec{y_1}$$

$$= (U\vec{x_1})^T (U\vec{x_2}) \qquad (AB)^T = B^T A^T$$

$$= \vec{x_1}^T \underbrace{\cancel{U^T U}}_{I} \vec{x_2}$$

$$= \vec{x_1}^T \vec{x_2} = \langle \vec{x_1}, \vec{x_2} \rangle$$

inner product is preserved in the new U basis

(b) $\|\vec{y_1}\|^2$, $\|\vec{y_2}\|^2$ $\quad \|\vec{y_1}\|_2^2$

assume it is 2 if no subscript

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \dots x_n^2}$$

$\vec{x} \in \mathbb{R}^N$

$$\|\vec{x}\|_2^2 = x_1^2 + x_2^2 + \dots x_n^2$$

<u>Hint</u> : Think about how you can express $\|\vec{x}\|_2^2$ as an inner product

$$\vec{x}^T \vec{x} = x_1^2 + x_2^2 + \dots x_n^2$$

$$\boxed{\|\vec{x}\|_2^2 = \vec{x}^T \vec{x} = \langle \vec{x}, \vec{x} \rangle}$$

Holds for real-valued vectors

$$\|\vec{y_1}\|_2^2 = \|U\vec{x_1}\|_2^2$$

$$= (U\vec{x_1})^T (U\vec{x_1})$$

$$= \vec{x_1}^T \cancel{U^T U} \vec{x_1} = \vec{x_1}^T \vec{x_1} = \|\vec{x_1}\|_2^2$$

norm is preserved

→ Same thing for $\|\vec{y_2}\|_2^2$

(c) $y_i = a^T \vec{x}_i \rightarrow \begin{matrix} y_1 = \vec{a}^T \vec{x}_1 \\ y_2 = \vec{a}^T \vec{x}_2 \\ \vdots \\ y_m = \vec{a}^T \vec{x}_m \end{matrix}$

Goal: $\vec{y} = X\vec{a}$

not quite

$$\therefore \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \vec{a}^T \begin{bmatrix} \vec{x}_1 \\ \vdots \\ \vec{x}_m \end{bmatrix}$$

$\vec{a}^T \vec{x}_i$ is a scalar
$\vec{x}_i^T \vec{a}$ has to be the same

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} \vec{x}_1^T \\ \vdots \\ \vec{x}_m^T \end{bmatrix}}_{X} \vec{a}$$

$$X = \begin{bmatrix} \vec{x}_1^T \\ \vdots \\ \vec{x}_m^T \end{bmatrix}$$

To estimate for $\vec{a}$ use least squares:

$$\therefore \hat{\vec{a}} = (X^T X)^{-1} X^T \vec{y}$$

$\text{rank}(X) = \text{rank}(X^T X)$

only invertible if full rank

X has to be full rank
$\longrightarrow$ X has linearly independent columns

(d) $X = MV^T$

$\rightarrow$ plug in $X = MV^T$ into and simplify

$$\therefore \hat{\vec{a}} = ((MV^T)^T MV^T)^{-1} (MV^T)^T \vec{y}$$
$$= (VM^T MV^T)^{-1} VM^T \vec{y}$$
$$= (V^T)^{-1} (M^T M)^{-1} (V)^{-1} VM^T \vec{y}$$
$$= V (M^T M)^{-1} \underbrace{\cancel{V^T V}}_{1} M^T \vec{y}$$
$$= V (M^T M)^{-1} M^T \vec{y}$$

$(AB)^T = B^T A^T$

$(AB)^{-1} = B^{-1} A^{-1} \rightarrow (ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

$V^T = V^{-1}$

$(\underbrace{V}_{A} \underbrace{M^T M}_{B} \underbrace{V^T}_{C})^{-1} = (V^T)^{-1} (M^T M)^{-1} (V)^{-1}$

(e) $X = \underbrace{U}_{} \Sigma \underbrace{V^T}_{}$ [SVD of a matrix]

orthonormal matrices

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ \hline & 0 & \end{bmatrix}$$

Previously $X = MV^T$

$X = U\Sigma V^T$ $\quad M = U\Sigma$

$\therefore$ Plug $M = U\Sigma$ to solution in (d)

$$\Sigma^T = \left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n & \end{array}\right]$$

From (d) $\hat{\vec{a}} = V(M^T M)^{-1} M^T \vec{y}$

$$= V\left((U\Sigma)^T (U\Sigma)\right)^{-1} (U\Sigma)^T \vec{y}$$

$$= V(\Sigma^T \cancel{U^T U} \Sigma)^{-1} \Sigma^T U^T \vec{y}$$

$$= V(\Sigma^T \Sigma)^{-1} \Sigma^T U^T \vec{y}$$

$$\Sigma^T \Sigma = \left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n & \end{array}\right] \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ \hline & 0 & \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & 0 & & 0 \\ 0 & \sigma_2^2 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \sigma_n^2 \end{bmatrix}$$

$\sigma_1^2 + 0 + 0 \dots + 0$

$$(\Sigma^T \Sigma)^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_n^2} \end{bmatrix}$$

$$\therefore \hat{\vec{a}} = V \begin{bmatrix} \frac{1}{\sigma_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_n^2} \end{bmatrix} \left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n & \end{array}\right] U^T \vec{y}$$

$$= V \left[\begin{array}{ccc|ccc} \frac{1}{\sigma_1} & & 0 & 0 & 0 & 0 \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & & \frac{1}{\sigma_n} & 0 & 0 & 0 \end{array}\right] U^T \vec{y}$$

$$= V \left[\begin{array}{ccc|c} \frac{1}{\sigma_1} & & 0 & \\ & \ddots & & 0 \\ 0 & & \frac{1}{\sigma_n} & \end{array}\right] U^T \vec{y}$$