To do: Minimum Energy Control

Recall: $x[i+1] = ax[i] + b\underline{u[i]}$

no contraint on $u[i]$

Today: Minimize Energy $\|\vec{u}\|_2^2 = u_1^2 + u_2^2 + \dots + u_n^2$

$\downarrow$

Recall: $x[1] = ax[0] + bu[0] = bu[0]$ (assume $x[0]=0$)

(a) $x[2] = abu[0] + bu[1]$

Think back to disc 2A/6A $x[3] = a^2 b u[0] + abu[1] + bu[2]$

$\vdots$

$$x[\ell] = a^{\ell-1} bu[0] + a^{\ell-2} bu[1] + \dots + abu[\ell-2] + bu[\ell-1]$$

$$= \underbrace{[a^{\ell-1}b, a^{\ell-2}, \dots, ab, b]}_{\text{fixed (given we know a and b)}} \underbrace{\begin{bmatrix} u[0] \\ \vdots \\ u[\ell-1] \end{bmatrix}}_{u \text{ vector isolated}}$$

plug in $\ell=10$

(b) $x[i+1] = 1.0x[i] + 0.7u[i]$, $x[0]=0$, $x[10]=14$

$$x[10] = 14 = a^9 bu[0] + a^8 bu[1] + \dots + bu[9]$$

$$14 = 0.7u[0] + 0.7u[1] + \dots + 0.7u[9]$$

$$20 = u[0] + u[1] + \dots + u[9]$$

Simplify: $20 = u[0] + u[1]$

Pick $u[0] = 15$, $u[1] = 5 \rightarrow$ Cost $= 15^2 + 5^2 = 250$

decreased

Pick $u[0] = 10$, $u[1] = 10 \rightarrow$ Cost $= 10^2 + 10^2 = 200$

Extend Symmetry argument:

$20 = u[0] + u[1] + \dots + u[9] \rightarrow$ Pick $u[0] = u[1] = \dots = u[9] = 2$

(c) $x[i+1] = 0.5x[i] + 0.7u[i]$, $x[0]=0$, $x[10]=14$

Intuitively : If we apply at the start, $u[0]$ will be halved at every time step $\Rightarrow$ to reach $x[10]=14$, we need a very large input

- If we apply at the end, $u[9]$ will be halved $\Rightarrow$ to reach $x[10]=14$, we need a MUCH smaller input

$\downarrow\downarrow$ cost

$\therefore$ Better to apply at the end.

(d) $x[i+1] = 0.5x[i] + 0.7u[i]$, $x[0]=0$, $x[10]=14$

Apply $u[8]$ and $u[9]$ Goal : Minimize $u[8]^2 + u[9]^2$

$x[9] = \cancel{0.5x[8]}\,0 + 0.7u[8]$
$x[10] = 0.5x[9] + 0.7u[9]$

plug in $\therefore x[10] = 0.5(0.7u[8]) + 0.7u[9]$

$14 = 0.35u[8] + 0.7u[9]$

$14 = 0.35u[8] + 0.7u[9]$

$u[8] = \dfrac{14 - 0.7u[9]}{0.35} = 40 - 2u[9]$

$\therefore$ Minimize $(40-2u[9])^2 + u[9]^2$

$= 5u[9]^2 - 160u[9] + 1600$

find $u[9]$ which minimizes the function

$\therefore \dfrac{d}{du[9]}\left(5u[9]^2 - 160u[9] + 1600\right) = 10u[9] - 160$

$\therefore$ Minimum when derivative $= 0$

$\therefore\ 10u[9] - 160 = 0 \Rightarrow u[9] = 16$, $\therefore u[8] = 40 - 2(16) = 8$

$\therefore$ Pick $u[8] = 8$, $u[9] = 16$

(f) $\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]$

(i) $\vec{x}[i+1] = \begin{array}{c} 1 \\ 2 \end{array}\overbrace{\begin{bmatrix} & \\ & \end{bmatrix}} \begin{bmatrix} u[0] \\ \vdots \\ u[i-1] \end{bmatrix}$

in part (a) $x[l] = a^{l-1} b u[0] + a^{l-2} b u[1] + \cdots + a b u[l-2] + b u[l-1]$

$= [a^{l-1} b, a^{l-2}, \ldots, ab, b] \begin{bmatrix} u[0] \\ \vdots \\ u[l-1] \end{bmatrix}$

(i)

but 2D same thing

$$\vec{x}[l] = \underline{[A^{l-1}\vec{b}, A^{l-2}\vec{b}, \ldots A\vec{b}, \vec{b}]} \begin{bmatrix} u[0] \\ \vdots \\ u[l-1] \end{bmatrix}$$

(ii) $[A^{l-1}\vec{b}, \ldots, \vec{b}]$

$= [A^{19}\vec{b}, A^{18}\vec{b}, \ldots, Ab, \vec{b}]$

$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^3 = A$

$A^4 = I$

if the power of A is even $\Rightarrow I$

if " " " " is odd $\Rightarrow A$

$\therefore [A^{19}\vec{b}, A^{18}\vec{b}, \ldots, Ab, \vec{b}]$

$= \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{bmatrix}$

$\therefore \vec{x}[20] = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} u[0] \\ \vdots \\ u[l-1] \end{bmatrix}$

(g) $\vec{x}[20] = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \rightarrow$ think about (b)

$\Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{pmatrix} u[0] + u[2] + u[4] + \cdots + u[l-2] \\ u[1] + u[3] + u[5] + \cdots + u[l-1] \end{pmatrix}$

$\rightarrow$ solving 2 optimization problems $\Rightarrow$ 1 each row

$\alpha = u[0] + u[2] + \dots + u[l-2] \rightarrow$ same problem as (b)!

$\therefore u[0] = u[2] = \dots = u[l-2] = \frac{\alpha}{10}$

10 terms of $u[i]$ in each row

Using the same argument for β

$\therefore u[1] = u[3] = \dots = u[l-1] = \frac{\beta}{10}$

$\therefore$ Pick $\vec{u} = \begin{bmatrix} u[0] \\ \vdots \\ u[l-1] \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{10} \\ \frac{\beta}{10} \\ \vdots \\ \frac{\alpha}{10} \\ \frac{\beta}{10} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} \alpha \\ \beta \\ \vdots \\ \alpha \\ \beta \end{bmatrix}$

(c) $x[i+1] = 0.5x[i] + 0.7u[i]$, $x[0] = 0$, $x[10] = 14$

$\downarrow$

$$14 = \vec{x}[10] = \underbrace{[0.5^9 \cdot 0.7,\ 0.5^8 \cdot 0.7,\ \dots,\ 0.5 \cdot 0.7,\ 0.7]}_{\vec{v}} \underbrace{\begin{bmatrix} u[0] \\ | \\ u[10] \end{bmatrix}}_{\vec{u}}$$

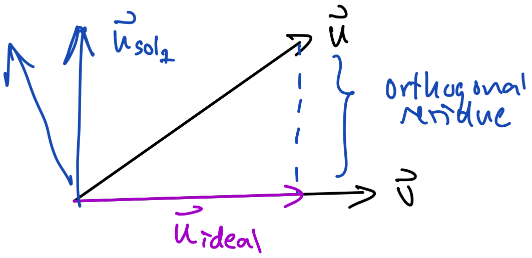
<u>Cauchy-Schwarz</u>: $\langle \vec{u}, \vec{v} \rangle \le \|\vec{u}\| \|\vec{v}\|$

We can write our problem as an inner product

$\because \vec{v}$ is a constant vector, $\therefore$ to minimize $\|\vec{u}\|_2^2$ is equivalent to minimizing $\langle \vec{u}, \vec{v} \rangle$

Minimum when $\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \|\vec{v}\|$

$\rightarrow$ minimum when $\vec{u}$ is in same direction as $\vec{v}$



$\langle \vec{u}, \vec{v} \rangle = 14$, let $\vec{u} = \alpha \vec{v}$

$\therefore \langle \alpha \vec{v}, \vec{v} \rangle = 14$

$\alpha \langle \vec{v}, \vec{v} \rangle = 14$

$\alpha (0.808)^2 = 14$

$$\alpha = \frac{14}{(0.808)^2} = 21.42$$

$\therefore$ Optimal $\vec{u} = 21.42 \vec{v}$

$$\langle \vec{u}, \underbrace{\vec{v}}_{\text{fixed}} \rangle \leq \|\vec{u}\|_2 \underbrace{\|\vec{v}\|_2}_{\text{fixed}}$$

$$14 = \langle \vec{u}, \vec{v} \rangle = \underline{\|\vec{u}\|_2} \|\vec{v}\|_2$$