To do: Linearization

Goal : Model nonlinear systems/functions as accurately as possible

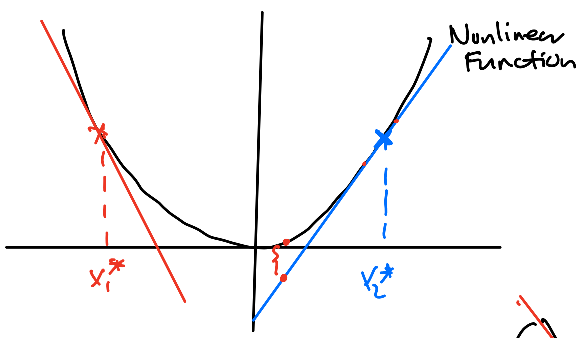
Recall : Taylor Expansion

$$f(x) \approx f(x^*) + \frac{f'(x^*)(x-x^*)}{1!} + \ldots$$

equilibrium point x^*

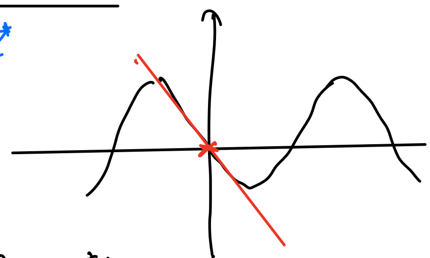
→ point in which we want to linearize our system around

→ something you can choose



E.g. $f(x) = -\sin(x)$,

$x^* = 0$



$\therefore f(x) \approx f(x^*) + f'(x^*)(x - x^*)$

$f(x) \approx 0 + -\cos(0)(x-0) \approx -x$

Sanity check : Linearizing a function, should always give a function back

1(a)(b) $f(x) = x^3 - 3x^2$

(a) $f(x) \approx f(x^*) + (3x^2 - 6x)\big|_{x=x^*}(x - x^*)$

(b) $f(x) \approx f(1.5) + (3 \cdot 1.5^2 - 6 \cdot 1.5)(x - 1.5)$

$f(x) \approx \underbrace{-3.375 + (-2.25)\cdot(x-1.5)}_{\text{linearized function}}$

vs.

$\hat{f}(x=1.7) = -3.825$

$f(x=1.7) = -3.757$

Fairly close! Pretty good approximation

$\hat{f}(x=2.5) = -5.625$

$f(x=2.5) = -3.125$

Pretty bad!

(c)

$$f(x) \approx f(x^*) + f'(x^*)(x-x^*)$$

$$f(x,y) \approx f(x^*,y^*) + \frac{\partial f}{\partial x}\bigg|_{x^*,y^*}(x-x^*) + \frac{\partial f}{\partial y}\bigg|_{x^*,y^*}(y-y^*)$$

partial derivative

Partial Derivative : $f(x,y) = 4x^3y^2$

$$\frac{\partial f}{\partial x} = 12x^2y^2$$

$$\frac{\partial f}{\partial y} = 8x^3y$$

(c), (d), (e) $f(x) = x^2y$

(c) $\frac{\partial f}{\partial x} = 2xy$

$\frac{\partial f}{\partial y} = x^2$

(d) $f(x,y) \approx f(x^*,y^*) + \frac{\partial f}{\partial x}\big|_{x^*,y^*}(x-x^*) + \frac{\partial f}{\partial y}\big|_{x^*,y^*}(y-y^*)$

$\approx f(x^*,y^*) + 2xy\big|_{x^*,y^*}(x-x^*) + x^2\big|_{x^*,y^*}(y-y^*)$

(e) $(x^*,y^*) = (2,3)$

$f(x,y) \approx 12 + 12(x-2) + 4(y-3)$

Approximation at $(2+\delta, 3+\delta)$

$\therefore$ Approx $= 12 + 16\delta$

Actual value at $(2+\delta, 3+\delta) = 12 + 16\delta + 7\delta^2 + \delta^3$

Error $= |\delta^3 + 7\delta^2|$

at $\delta = 0.01$, error $= 0.000701$

<u>Extend it further</u> : $f(\vec{x}) \rightarrow$ function that takes in $x_1 \dots x_n$ as your input

$$\vec{x} \in \mathbb{R}^N \rightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$f(\vec{x}) \approx f(\vec{x}^*) + \underbrace{\frac{\partial f}{\partial x_1}\Big|_{x_1^*}(x_1 - x_1^*) + \frac{\partial f}{\partial x_2}\Big|_{x_2^*}(x_2 - x_2^*) + \dots}_{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big|_{x_i^*}(x_i - x_i^*)}$$

dot product of 2 vector

$$\underbrace{\begin{bmatrix} \frac{\partial f}{\partial x_1}, & \frac{\partial f}{\partial x_2}, & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}}_{J_{\vec{x}} f} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \\ \vdots \\ x_n - x_n^* \end{bmatrix}$$

$$f(\vec{x}) \approx f(\vec{x}^*) + J_{\vec{x}} f \cdot (\vec{x} - \vec{x}^*)$$

$$f(\underset{\mathbb{R}^n}{\vec{x}}, \underset{\mathbb{R}^k}{\vec{y}}) \approx f(\vec{x}^*, \vec{y}^*) + \underset{n \text{ terms}}{J_{\vec{x}} f}(\vec{x} - \vec{x}^*) + \underset{k \text{ terms}}{J_{\vec{y}} f}(\vec{y} - \vec{y}^*)$$

(f) $f(\vec{x}, \vec{y}) = \vec{x}^T \vec{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

$$J_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} y_1, & y_2, & y_3, & \dots & y_n \end{bmatrix} = \vec{y}^T$$

$$J_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \dots & \frac{\partial f}{\partial y_n} \end{bmatrix} = \begin{bmatrix} x_1, & x_2, & \dots & x_n \end{bmatrix} = \vec{x}^T$$

(g) $f(\vec{x}, \vec{y}) = \vec{x}^T \vec{y}$, $\vec{x}^* = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{y}^* \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\approx f(\vec{x}^*, \vec{y}^*) + \vec{y}^T \Big|_{\vec{x}^*, \vec{y}^*}(\vec{x} - \vec{x}^*) + \vec{x}^T \Big|_{\vec{x}^*, \vec{y}^*}(\vec{y} - \vec{y}^*)$$

$$\approx 3 + \begin{bmatrix} -1 \\ 2 \end{bmatrix}^T \left(\vec{x} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \left(\vec{y} - \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$