To do: Linearization Part II

Recall: $f(\vec{x},\vec{y}) \approx f(\vec{x}^*,\vec{y}^*) + J_{\vec{x}} f\Big|_{\vec{x}^*,\vec{y}^*}(\vec{x}-\vec{x}^*) + J_{\vec{y}} f\Big|_{\vec{x}^*,\vec{y}^*}(\vec{y}-\vec{y}^*)$

$\in \mathbb{R}^n$ extension

$J_{\vec{x}} f \to \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$ $\quad J_{\vec{y}} f \to \begin{bmatrix} \frac{\partial f}{\partial y_1}, & \cdots, & \frac{\partial f}{\partial y_n} \end{bmatrix}$

$$f(x,y) \approx f(x^*,y^*) + \frac{\partial f}{\partial x}\Big|_*(x-x^*) + \frac{\partial f}{\partial y}\Big|_*(y-y^*)$$

Today: Vector of functions

$$\vec{f}(\vec{x},\vec{y}) = \begin{bmatrix} f_1(\vec{x},\vec{y}) \\ \vdots \\ f_m(\vec{x},\vec{y}) \end{bmatrix} \approx \begin{bmatrix} f_1(\vec{x}^*,\vec{y}^*) + J_{\vec{x}} f_1 \cdot (\vec{x}-\vec{x}^*) + J_{\vec{y}} f_1 (\vec{y}-\vec{y}^*) \\ \vdots \\ f_m(\vec{x}^*,\vec{y}^*) + J_{\vec{x}} f_m \cdot (\vec{x}-\vec{x}^*) + J_{\vec{y}} f_m (\vec{y}-\vec{y}^*) \end{bmatrix}$$

$$\vec{f}(\vec{x},\vec{y}) \approx \begin{bmatrix} f_1(\vec{x}^*,\vec{y}^*) \\ \vdots \\ f_m(\vec{x}^*,\vec{y}^*) \end{bmatrix} + \begin{bmatrix} J_{\vec{x}} f_1 \\ \vdots \\ J_{\vec{x}} f_m \end{bmatrix}(\vec{x}-\vec{x}^*) + \begin{bmatrix} J_{\vec{y}} f_1 \\ \vdots \\ J_{\vec{y}} f_m \end{bmatrix}(\vec{y}-\vec{y}^*)$$

$$\approx \vec{f}(\vec{x}^*,\vec{y}^*) + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial f_m}{\partial x_1} & \cdots & & \frac{\partial f_m}{\partial x_n} \end{bmatrix}}_{\text{Jacobian Matrix}} m \times n \; (\vec{x}-\vec{x}^*) + \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_n} \end{bmatrix}(\vec{y}-\vec{y}^*)$$

→ each row of the Jacobian matrix is 1 function

1(a) (c)

$$\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix} \begin{matrix} f_1 \\ f_2 \end{matrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$J_x \vec{f} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}}_{2\times 2} = \begin{bmatrix} 2x_1 x_2 & x_1^2 \\ x_2^2 & 2x_1 x_2 \end{bmatrix}$$

(c) $\vec{f}(\vec{x},\vec{y}) = \vec{x}\vec{y}^T\vec{w}$, $\vec{x},\vec{y},\vec{w}$ 2 rows

Hint: Write out the product of the vectors

$$\vec{f}(\vec{x},\vec{y}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 y_1 w_1 + x_1 y_2 w_2 \\ x_2 y_1 w_1 + x_2 y_2 w_2 \end{bmatrix} \begin{array}{l} \leftarrow f_1 \\ \leftarrow f_2 \end{array}$$

$$J_{\vec{x}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} y_1 w_1 + y_2 w_2 & 0 \\ 0 & y_1 w_1 + y_2 w_2 \end{bmatrix}$$

$$J_{\vec{y}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} x_1 w_1 & x_1 w_2 \\ x_2 w_1 & x_2 w_2 \end{bmatrix}$$

(b) $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$

$$J_x\vec{f} = \underset{2\times 2}{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}} = \begin{bmatrix} 2x_1 x_2 & x_1^2 \\ x_2^2 & 2x_1 x_2 \end{bmatrix}$$

$\vec{x}^* = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Actual value at $\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}$ is $\begin{bmatrix} 12.160701 \\ 18.210801 \end{bmatrix}$

vs.

Approximation is $\vec{f}(\vec{x}) \approx \vec{f}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) + \begin{bmatrix} 2x_1 x_2 & x_1^2 \\ x_2^2 & 2x_1 x_2 \end{bmatrix}\Bigg|_{\vec{x}^*} (\vec{x} - \vec{x}^*)$

$$\approx \vec{f}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) + \begin{bmatrix} 12 & 4 \\ 9 & 12 \end{bmatrix} \left(\vec{x} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$$

(approximation function)

at $\begin{bmatrix}2.01\\3.01\end{bmatrix}$, my approximation value is $\begin{bmatrix}12.16\\18.21\end{bmatrix}$

2. $$\frac{d}{dt}\underbrace{\begin{bmatrix}\beta(t)\\ \gamma(t)\end{bmatrix}}_{\vec{x}(t)} = \begin{bmatrix}-2\beta(t)+\gamma(t)\\ g(\gamma(t)) + u(t)\end{bmatrix}\begin{matrix} f_1 \\ f_2 \end{matrix} = \vec{f}(\vec{x}(t), u(t)) \leftarrow \text{non-linear}$$

Recall : $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$ [Linear]

$\vec{x}^*$ is an operating point if $\vec{f}(\vec{x}^*(t), u^*(t)) = \vec{0}$

$u^* = -1$, find $\vec{x}^*$ such that $\vec{f}(\vec{x}^*(t), u^*(t)) = 0$

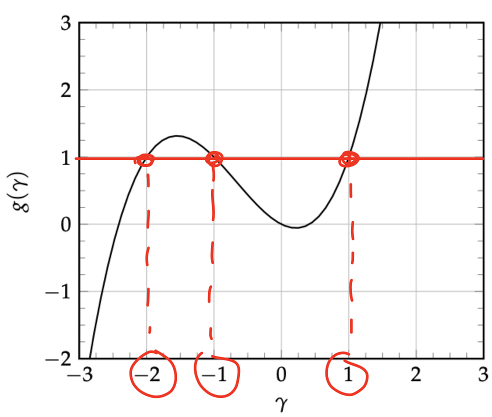
$$\begin{bmatrix}-2\beta(t)+\gamma(t)\\ \underbrace{g(\gamma(t))}_{1} + \underbrace{u(t)}_{-1}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad \text{where } \vec{x}^*(t) = \begin{bmatrix}\beta(t)\\ \gamma(t)\end{bmatrix}$$

$\therefore g(\gamma(t)) = 1$

$\gamma_1 = -2 \rightarrow \beta_1 = -1$

$\gamma_2 = -1 \rightarrow \beta_2 = -1/2$

$\gamma_3 = 1 \rightarrow \beta_3 = \frac{1}{2}$



$\therefore$ 3 equilibrium points

$$\begin{cases} \vec{x}_1^*(t) = \begin{bmatrix}-1\\-2\end{bmatrix} \\ \vec{x}_2^*(t) = \begin{bmatrix}-1/2\\-1\end{bmatrix} \\ \vec{x}_3^*(t) = \begin{bmatrix}1/2\\1\end{bmatrix}\end{cases}$$

(b) Linearize around $\vec{x}_3^* = \begin{array}{cc} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} & \begin{array}{c} \beta \\ \gamma \end{array} \end{array}$

$$\vec{f}(\vec{x}(t), u(t)) = \begin{array}{cc} \begin{bmatrix} -2\beta(t) + \gamma(t) \\ g(\gamma(t)) + u(t) \end{bmatrix} & \begin{array}{c} \leftarrow f_1 \\ \leftarrow f_2 \end{array} \end{array}$$

$$\therefore \vec{f}(\vec{x}_3^*(t), \underbrace{u^*(t)}_{=-1}) \approx \vec{f}\left(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, -1\right) + J_{\vec{x}}\vec{f}\,\Big|_{\vec{x}_3^*, u^*} (\vec{x} - \vec{x}_3^*)$$

$$+ J_u\vec{f}\,\Big|_{\vec{x}_3^*, u^*} (u - u^*)$$

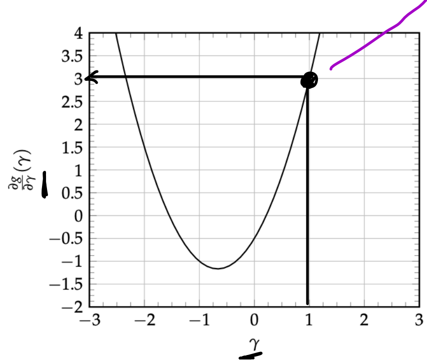
$$J_{\vec{x}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial \beta} & \frac{\partial f_1}{\partial \gamma} \\ \frac{\partial f_2}{\partial \beta} & \frac{\partial f_2}{\partial \gamma} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & \frac{\partial g}{\partial \gamma} \end{bmatrix}$$

$$J_u\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\approx \underbrace{\vec{f}\left(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, -1\right)}_{0} + \underbrace{\begin{bmatrix} -2 & 1 \\ 0 & \frac{\partial g}{\partial \gamma} \end{bmatrix}}_{A_3}\Big|_{\vec{x}^*, u^*} \overbrace{\cancel{(\vec{x} - \vec{x}_3^*)}}^{\delta \vec{x}_3^*} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_3} \delta u(t)$$

Evaluating $\frac{\partial g}{\partial \gamma}$ at $\gamma = 1 \implies 3$

$$\therefore A_3 = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$$



(c) $$\underbrace{\frac{d}{dt}\begin{bmatrix} \beta(t) \\ \gamma(t) \end{bmatrix}}_{\vec{x}(t)} = \begin{array}{cc} \begin{bmatrix} -2\beta(t) + \gamma(t) \\ g(\gamma(t)) + u(t) \end{bmatrix} & \begin{array}{c} f_1 \\ f_2 \end{array} \end{array}$$

Recall : $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$ [Linear]

$$\frac{d}{dt}\begin{bmatrix} p(t) \\ \gamma(t) \end{bmatrix} = \begin{bmatrix} -2p(t) + \gamma(t) \\ g(\gamma(t)) + u(t) \end{bmatrix}$$

$$(\text{at } \vec{x}_3^*) \approx \cancel{\vec{f}\left(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, -1\right)} \to 0 + \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}_{\vec{x}^*, u^*} \delta\vec{x}_3(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u(t)$$

$$\approx \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix} \delta \vec{x}_3(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u(t)$$

approximated linear CT system at $\vec{x}_3^*$

(c) $\vec{x}_3^*$ is not stable because eigenvalues are -2 and 3

$$f(x) \approx f(x^*) + \underline{f'(x^*)(x - x^*)}$$

- taylor expansion gave us linear approx. of $f(x)$

$\downarrow$

$$f(x,y) \approx f(x^*, y^*) + \underline{\left.\frac{\partial f}{\partial x}\right|_{x^*, y^*}(x - x^*)} + \left.\frac{\partial f}{\partial y}\right|_{x^*, y^*}(y - y^*)$$

$\downarrow$

$f(\vec{x})$

$A^T A \rightarrow$ eigenvectors forms V basis

$A A^T \rightarrow$ eigenvectors forms U basis

Frobenius form $\rightarrow$ low rank approx