Complex Inner Products

Recall: $\langle \vec{x},\vec{y}\rangle = \vec{y}^T\vec{x}$ if $\vec{x},\vec{y}\in\mathbb{R}^n$

$\rightarrow \langle \vec{x},\vec{y}\rangle = \langle \vec{y},\vec{x}\rangle$

not true for complex case

Today: $\vec{u},\vec{v}\in\mathbb{C}^n$

$$\langle \vec{u},\vec{v}\rangle = \vec{v}^{*}\vec{u}$$

conjugate transpose

$$= \sum_{i=1}^{n} \bar{v}_i \cdot u_i$$

- $\|\vec{u}\| = \sqrt{\langle \vec{u},\vec{u}\rangle}$ is always real

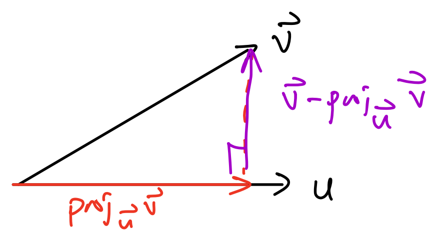
(a) $\text{proj}_{\vec{u}}(\alpha\vec{u}) = \alpha\vec{u}$

$$\text{proj}_{\vec{u}}(\alpha\vec{u}) = \frac{\langle \alpha\vec{u},\vec{u}\rangle}{\langle \vec{u},\vec{u}\rangle}\cdot\vec{u}$$

$$= \frac{\vec{u}^{*}(\alpha\vec{u})}{\vec{u}^{*}\vec{u}}\cdot\vec{u}$$

$$= \frac{\alpha\,\cancel{(\vec{u}^{*}\vec{u})}}{\cancel{\vec{u}^{*}\vec{u}}}\cdot\vec{u} = \alpha\vec{u} \checkmark$$

(b) $\langle \vec{u},\vec{v}\rangle = \vec{v}^{*}\vec{u} = 0$



Show that $\langle \vec{u}, \vec{v}-\text{proj}_u\vec{v}\rangle = 0$

$$= (\vec{v}-\text{proj}_{\vec{u}}\vec{v})^{*}\vec{u}$$

$$= \left(\vec{v}-\left(\left(\frac{\langle \vec{v},\vec{u}\rangle}{\langle \vec{u},\vec{u}\rangle}\right)\vec{u}\right)\right)^{*}\vec{u}$$

$$= \vec{v}^{*}\vec{u} - \left(\left(\frac{\langle \vec{v},\vec{u}\rangle}{\langle \vec{u},\vec{u}\rangle}\right)\vec{u}\right)^{*}\vec{u}$$

$$(A-B)^T = A^T - B^T$$

$$(AB)^T = B^T A^T$$

$$\vec{u}^{*}\vec{u} = \bar{\vec{u}}^T\vec{u}$$

$$= \vec{v}^* \vec{u} - \left(\frac{\langle \vec{v}, \vec{u}\rangle}{\langle \vec{u}, \vec{u}\rangle}\right)^* \vec{u}^* \vec{u}$$

$$= \vec{v}^* \vec{u} - \frac{(\langle \vec{v}, \vec{u}\rangle)^*}{(\langle \vec{u}, \vec{u}\rangle)^*} \cdot \vec{u}^* \vec{u}$$

$\langle \vec{u}, \vec{u}\rangle \in \mathbb{R}$

$\therefore (\langle \vec{u}, \vec{u}\rangle)^* = \langle \vec{u}, \vec{u}\rangle$

$$= \vec{v}^* \vec{u} - (\langle \vec{v}, \vec{u}\rangle)^* \cdot \cancel{\frac{\vec{u}^* \vec{u}}{\langle \vec{u}, \vec{u}\rangle}}$$

$(\vec{x}^T \vec{y})^T = \vec{y}^T \vec{x}$

$$= \vec{v}^* \vec{u} - (\vec{u}^* \vec{v})^* = \vec{v}^* \vec{u} - \vec{v}^* \vec{u} = 0$$

Generally for $\vec{u}, \vec{v} \in \mathbb{C}^n$, $\langle \vec{u}, \vec{v}\rangle \neq \langle \vec{v}, \vec{u}\rangle$

$\rightarrow$ Correct : $\langle \vec{u}, \vec{v}\rangle = (\langle \vec{v}, \vec{u}\rangle)^*$

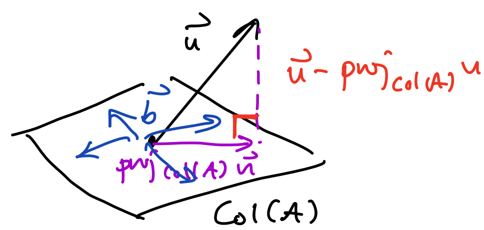
$$\langle \vec{v} - \text{proj}_{\vec{u}}(\vec{v}), \vec{u}\rangle = (\langle \vec{u}, \vec{v} - \text{proj}_{\vec{u}}(\vec{v})\rangle)^*$$
$$= (0)^* = 0$$

Orthogonal case doesn't matter but generally order matters!

(c) $\text{proj}_{Col(A)}(\vec{u}) = A(A^*A)^{-1}A^*\vec{u}$

Show that $\vec{u} = A\vec{x}$, $\vec{x} \in \mathbb{C}^n$, $\text{proj}_{Col(A)}(\vec{u}) = \vec{u}$

$$\text{proj}_{Col(A)}\vec{u} = A(A^*A)^{-1}A^*\vec{u}$$
$$= A\underbrace{(A^*A)^{-1}A^*(A}_{I_{n\times n}}\vec{x})$$
$$= A\vec{x}$$
$$= \vec{u}$$



(d) Show that $\langle \vec{u} - \text{proj}_{Col(A)}(\vec{u}), \vec{b}\rangle = 0$ if $\vec{b} \in Col(A)$

$$\langle \vec{u} - \text{proj}_{Col(A)}\vec{u}, \vec{b}\rangle = \vec{b}^*(\vec{u} - \text{proj}_{Col(A)}(\vec{u}))$$
$$= (A\vec{x})^*(\vec{u} - \text{proj}_{Col(A)}(\vec{u}))$$

$(A\vec{x})^T = \vec{x}^T A^T$

$$= (\vec{x}^* A^*)(\vec{u} - \text{proj}_{Col(A)} \vec{u})$$

$$= \vec{x}^* A^* \vec{u} - \vec{x}^* \underbrace{A^* A (A^* A)^{-1}}_{I_{n\times n}} A^* \vec{u}$$

$$= \vec{x}^* A^* \vec{u} - \vec{x}^* A^* \vec{u} = 0 \checkmark$$

(e) A is orthonormal if $\begin{cases} \langle \vec{a_i}, \vec{a_j} \rangle = \vec{a_j}^* \vec{a_i} = 0 \\ \|\vec{a_i}\| = \sqrt{\langle \vec{a_i}, \vec{a_i} \rangle} = \sqrt{\vec{a_i}^* \vec{a_i}} = 1 \end{cases}$

$$A = [\vec{a_1} \; \cdots \; \vec{a_n}]$$

$$A^* = \begin{bmatrix} \vec{a_1}^* \\ \vdots \\ \vec{a_n}^* \end{bmatrix} \longleftrightarrow A^T = \begin{bmatrix} \vec{a_1}^T \\ \vdots \\ \vec{a_n}^T \end{bmatrix}$$

$$\langle \vec{a_1}, \vec{a_1} \rangle \in \mathbb{R} = \|\vec{a_1}\|^2$$

$$A^* A = \begin{bmatrix} \vec{a_1}^* \\ \vdots \\ \vec{a_n}^* \end{bmatrix} [\vec{a_1} \cdots \vec{a_n}] = \begin{bmatrix} \vec{a_1}^* \vec{a_1} & \cdots & \vec{a_1}^* \vec{a_n} \\ \vdots & \ddots & \vdots \\ \vec{a_n}^* \vec{a_1} & \cdots & \vec{a_n}^* \vec{a_n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cdots & 0 \\ & 1 & \\ & & \ddots \\ 0 & & 1 \end{bmatrix} = I_{n\times n} \checkmark$$

Projection formula from (c) is

$$\text{proj}_{Col(A)}(\vec{u}) = A \underbrace{(A^* A)^{-1}}_{I_{n\times n}} A^* \vec{u}$$

$$= A A^* \vec{u}$$