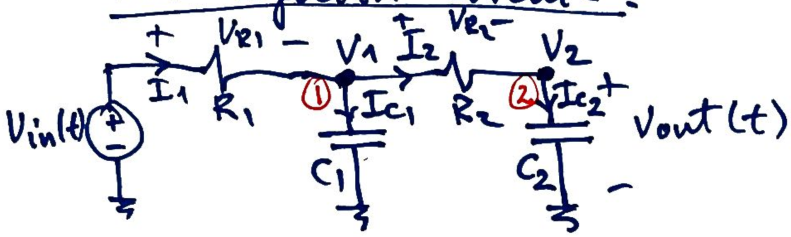
EECS 16B

Lecture 5

* Systems of differential equations
  * Higher-order diff. eqn. systems
  * Circuits w. multiple C's
  * Vector diff. eqns.
* Diagonalization

Circuits example – a more complex system:

Two-capacitor circuit:



KCL: $I_2 = I_{C_2}$

$I_1 = I_2 + I_{C_1}$

Elements: $I_{C_1} = C_1 \frac{dV_1}{dt}$

$I_{C_2} = C_2 \frac{dV_2}{dt}$

$V_{R_1} = I_1 \cdot R_1$

$V_{R_2} = I_2 \cdot R_2$

Voltages: $V_{in} - V_1 = V_{R_1}$

$V_{out} = V_2 - 0 = V_2$

$V_1 - V_2 = V_{R_2}$

From NVA directly:

① $$\frac{V_{in} - V_1}{R_1} = C_1 \frac{dV_1}{dt} + \frac{V_1 - V_2}{R_2}$$

② $$\frac{V_1 - V_2}{R_2} = C_2 \frac{dV_2}{dt} \Rightarrow \boxed{V_1 = V_2 + R_2 C_2 \frac{dV_2}{dt}}$$

$$\boxed{R_1 C_1 R_2 C_2 \frac{d^2 V_2}{dt^2} + (R_1 C_1 + R_1 C_2 + R_2 C_2)\frac{dV_2}{dt} + V_2 - V_{in} = 0}$$

(2^{nd} order diff. eq) – not yet know how to solve it.

Back to our system:

(1) $$\frac{V_{in}-V_1}{R_1} = C_1\frac{dV_1}{dt} + \frac{V_1-V_2}{R_2}$$

(2) $$\frac{V_1-V_2}{R_2} = C_2\frac{dV_2}{dt}$$

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(1) $$\frac{dV_1}{dt} = -\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}\right)V_1 + \frac{V_2}{R_2C_1} + \frac{V_{in}}{R_1C_1}$$

(2) $$\frac{dV_2}{dt} = \frac{V_1}{R_2C_2} - \frac{V_2}{R_2C_2}$$

In matrix-vector form

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} V_{in}$$

only know how to solve:

$$\frac{d}{dt}x(t) = \lambda x(t) + b u(t)$$

Example: Assume: $R_1 = \frac{1}{3} M\Omega$ ($M = 10^6$), $R_2 = \frac{1}{2} M\Omega$

$C_1 = C_2 = 1 \mu F$ ($\mu = 10^{-6}$)

(a) $$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in}$$

solutions for $t \geq 0$.

Assume:
$V_{in} = 1V$ $t<0$
$\downarrow$
$0V$ $t \geq 0$
$V_1(0) = 1V$
$V_2(0) = 1V$

"Magic" change of variables:

(b) $u_1 = V_2$

(c) $u_2 = V_1 + 2V_2$

$$\frac{d}{dt}u_1 \overset{(b)}{=} \frac{d}{dt}V_2 \overset{(a)}{=} 2V_1 - 2V_2 \overset{(c)}{=} 2(u_2 - 2V_2) - 2V_2 =$$

$$= 2u_2 - 4V_2 - 2V_2 = 2u_2 - 6V_2 \overset{(b)}{=} 2u_2 - 6u_1$$

(d) $$\boxed{\frac{d}{dt}u_1 = \cancel{2u_2 - 6u_1} \; -6u_1 + 2u_2}$$

$$\frac{d}{dt}u_2 \overset{(c)}{=} \frac{d}{dt}(V_1 + 2V_2) = \frac{d}{dt}V_1 + 2\frac{d}{dt}V_2 \overset{(a)}{=} -5V_1 + 2V_2 +$$

$$+ 2(2V_1 - 2V_2) =$$

$$= -5V_1 + 2V_2 + 4V_1 - 4V_2 =$$

$$= -V_1 - 2V_2 \overset{(c)}{=} -u_2 \qquad \text{Wohoo!}$$

(e) $$\boxed{\frac{d}{dt}u_2 = -u_2}$$ (homogeneous scalar diff-eq) $\Rightarrow$ know how to solve for u_2

$$u_2(t) = u_2(0)\,e^{-t}, \quad t \geq 0$$

$$u_2(0) \overset{(c)}{=} V_1(0) + 2V_2(0) = 1V + 2 \cdot 1V = 3V$$

(24)

(f) $\boxed{u_2(t) = 3 \cdot e^{-t}, \quad t \geq 0}$

Now, go back to (d)

$$\frac{d}{dt} u_1 = -6u_1 + 2u_2$$

$$\boxed{\frac{d}{dt} u_1 = -6u_1 + 6e^{-t}}, \quad t \geq 0$$

$\lambda = -6$

$s = -1$

Remember (lecture 4)

$$\frac{d}{dt} x(t) = \lambda x(t) - \lambda u(t)$$

$$u(t) = e^{st}$$

$$x(t) = K_2 e^{\lambda t} - \frac{\lambda}{s-\lambda} e^{st}$$

$$x(0) = K_2 - \frac{\lambda}{s-\lambda}$$

$$u_1(t) = K_2 e^{-6t} - \frac{-6 \cdot e^{-t}}{-1-(-6)}$$

$$u_1(t) = K_2 e^{-6t} + \frac{6}{5} e^{-t}$$

$$u_1(0) = K_2 + \frac{6}{5}$$

$$u_1(0) \overset{(6)}{=} V_2(0) = 1V \Rightarrow K_2 = 1 - \frac{6}{5} = -\frac{1}{5}$$

(g) $\boxed{u_1(t) = -\frac{1}{5} e^{-6t} + \frac{6}{5} e^{-t}}$

so, we solved for $u_1(t)$ & $u_2(t)$ $\Rightarrow$

$\Rightarrow$ back-solve for $V_1(t)$ & $V_2(t)$

$$V_1(t) \overset{(6)\&(C)}{=\!=} u_2(t) - 2u_1(t) \overset{(f)+(g)}{=\!=} 3e^{-t} - 2\left(-\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}\right)$$

$$= 3e^{-t} + \frac{2}{5}e^{-6t} - \frac{12}{5}e^{-t}$$

(h) $\boxed{V_1(t) = \frac{3}{5}e^{-t} + \frac{2}{5}e^{-6t}}$

(i) $V_2(t) \overset{(b)}{=} u_1(t) \overset{(g)}{=} \frac{6}{5} e^{-t} - \frac{1}{5} e^{-6t}$

(a) $$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\vec{b}} V_{in}$$

(b) & (c) $$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{W^{-1}} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}}_{W} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (j)$$

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{d}{dt}\left(W^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \right) = W^{-1} \frac{d}{dt} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \overset{(a)}{=}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{W^{-1}} \left(\underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\vec{b}} V_{in} \right)$$

$$\overset{(j)}{=} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{W^{-1}} \underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}}_{W} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{W^{-1}} \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\vec{b}} V_{in}$$

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -6 & 2 \\ 0 & -1 \end{bmatrix}}_{W^{-1}AW} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{W^{-1}\vec{b}} V_{in}$$

upper-triangular, so we can start at the bottom, solve & peel back.

Summary: Systems of diff. eqns.

(1) $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$, $\vec{x}(0)$, $\vec{x}(t) = ?$
$t \geq 0$.

Variables — Derivatives

Native $\vec{x}$ coordinates

"Nice" coordinates $\tilde{\vec{x}}$:

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \arrow[d, bend right, "V^{-1}"'] \\ \tilde{\vec{x}} \arrow[u, bend right, "V"'] \arrow[r, "?", "V^{-1}AV"'] & \frac{d}{dt}\tilde{\vec{x}} \arrow[u, bend right, "V"'] \end{tikzcd}$$

(2) $\vec{x}(t) = V\tilde{\vec{x}}(t)$

(3) $\tilde{\vec{x}}(t) = V^{-1}\vec{x}(t)$

$$\frac{d}{dt}\tilde{\vec{x}}(t) \overset{(3)}{=} \frac{d}{dt}\left(V^{-1}\vec{x}(t)\right) = V^{-1}\frac{d}{dt}\vec{x}(t) = V^{-1}\left(A\vec{x}(t) + B\vec{u}(t)\right)$$

$$= V^{-1}A\vec{x}(t) + V^{-1}B\vec{u}(t) \overset{(2)}{=}$$

$$= V^{-1}A\left(V\tilde{\vec{x}}(t)\right) + V^{-1}B\vec{u}(t)$$

(4) $$\boxed{\frac{d}{dt}\tilde{\vec{x}}(t) = \underbrace{V^{-1}AV}\tilde{\vec{x}}(t) + V^{-1}B\vec{u}(t)}$$

want this matrix to be "nice"

e.g. $\begin{bmatrix} \bullet & \\ 0 & \bullet \end{bmatrix}$ or even better $\begin{bmatrix} \bullet & & 0 \\ & \ddots & \\ 0 & & \bullet \end{bmatrix}$ (separable) homogeneous in $\tilde{x}_{i,j,k}$

(5) Don't forget:
$\tilde{\vec{x}}(0) = V^{-1}\vec{x}(0)$

(6) Go back to $\vec{x}(t)$: (2) $\vec{x}(t) = V\tilde{\vec{x}}(t)$