Lecture 8

* Solving systems of diff. eqns. with **Phasors**

* i.e. transforming systems of diff. eqs. into systems of linear eqns.

Refresher:

$$\frac{d}{dt}x(t) = \lambda x(t) + b u(t) \quad , \text{ when } u(t) = K \cdot e^{st}, \; s \neq \lambda$$

$t \geq 0$:

$$x(t) = \underbrace{\left(x(0) - \frac{bK}{s-\lambda}\right)e^{\lambda t}}_{\substack{\text{annoying term} \\ \text{transient solution} \\ \text{b/c of initial conditions}}} + \underbrace{\frac{bK}{s-\lambda}e^{st}}_{\substack{\text{Nice term, same form} \\ \text{as input (steady-state} \\ \text{solution)}}}$$

Want the transient (first) part to disappear for $t \to \infty$

If $\lambda < 0$, $e^{\lambda t} \xrightarrow[t\to\infty]{} 0$ and $x(t) \xrightarrow[t\to\infty]{} \underbrace{\frac{bK}{s-\lambda}e^{st}}_{\substack{\text{steady-state} \\ \text{solution}}}$

What about complex λ?

$$e^{\lambda t} = e^{(\lambda_r + j\lambda_i)t} = e^{\lambda_r t} \cdot e^{j\lambda_i t} = e^{\lambda_r t}(\cos(\lambda_i t) + j\sin(\lambda_i t))$$

so if $\lambda_r < 0$ $\quad e^{\lambda_r t} \xrightarrow[t\to\infty]{} 0$ & $x(t) \xrightarrow[t\to\infty]{} \frac{bK}{s-\lambda}e^{st}$

(steady-state solution)

(1) $$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t)$$ system of diff. eqns.

Let's consider inputs of the form $\sim e^{st}$ and assert that the solutions are $\sim e^{st}$, valid $s \neq \lambda$ and $Re(\lambda) < 0$, in steady-state.

$$\vec{u}(t) = \vec{\tilde{u}} e^{st}, \quad \vec{\tilde{u}} \text{ vector of constants}$$

Assert: $\vec{x}(t) = \vec{\tilde{x}} e^{st}$, $\vec{\tilde{x}}$ vector of constants

steady-state soln.

$$\frac{d}{dt}\vec{x}(t) = \vec{\tilde{x}} \frac{d}{dt} e^{st} = s\vec{\tilde{x}} e^{st}$$

$$s\vec{\tilde{x}} \cancel{e^{st}} = A\vec{\tilde{x}} \cancel{e^{st}} + \vec{\tilde{u}} \cancel{e^{st}}$$

$$s\vec{\tilde{x}} = A\vec{\tilde{x}} + \vec{\tilde{u}}$$

(2)
$$\boxed{\boxed{(sI - A)\vec{\tilde{x}} = \vec{\tilde{u}}} \quad \vec{\tilde{x}} = (sI - A)^{-1}\vec{\tilde{u}}}$$

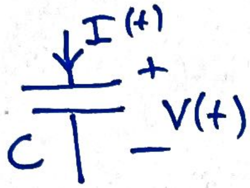
system of lin. equations

[Remember: $s \neq \lambda$ so $sI - A$ has no nullspace & is therefor invertible]

$$\vec{x}(t) = (sI - A)^{-1}\vec{\tilde{u}} e^{st}$$

solution to (1) for $\vec{u}(t) = \vec{\tilde{u}} e^{st}$, under $s \neq \lambda$ & $Re(\lambda) < 0$.

Can we use this for ckt directly?



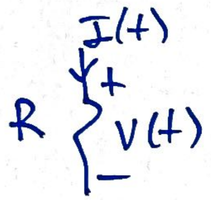
$$I(t) = C\frac{d}{dt}V(t)$$

Assert: $\left.\begin{array}{l} I(t) = \hat{I} e^{st} \\ V(t) = \hat{V} e^{st} \end{array}\right\} \quad \frac{V(t)}{I(t)} = \frac{\hat{V}}{\hat{I}}$

$$I(t) = \hat{I} e^{st} = C\frac{d}{dt}(\hat{V}e^{st}) = C\hat{V}\frac{d}{dt}(e^{st}) = sC\hat{V}e^{st}$$

$$\hat{I}\cancel{e^{st}} = sC\hat{V}\cancel{e^{st}}$$

$$\boxed{\frac{\hat{V}}{\hat{I}} = \frac{1}{sC}}$$ (capacitor s-impedance)



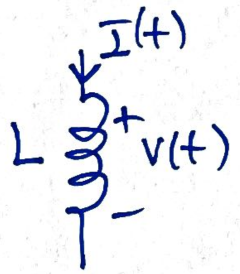
$V(t) = \hat{V}e^{st}$

$I(t) = \hat{I}e^{st}$

$V(t) = RI(t)$

$\hat{V}\cancel{e^{st}} = R\hat{I}\cancel{e^{st}}$

$$\boxed{\frac{\hat{V}}{\hat{I}} = R}$$ (resistor s-impedance)



$V(t) = \hat{V}e^{st}$

$I(t) = \hat{I}e^{st}$

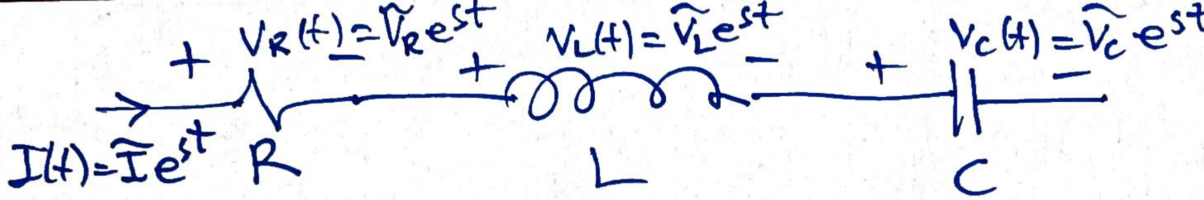
$V(t) = L\frac{d}{dt}I(t)$

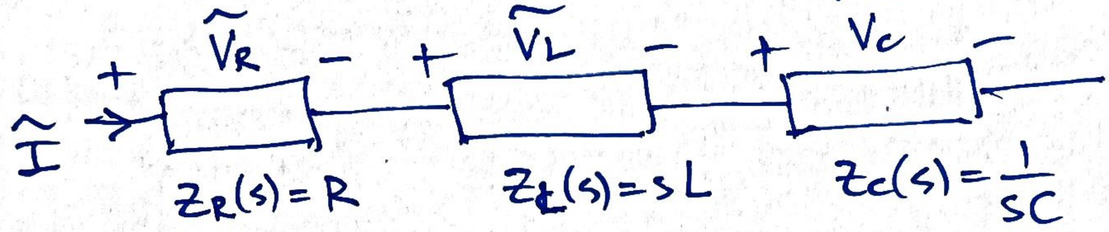
$\hat{V}e^{st} = L\frac{d}{dt}(\hat{I}e^{st})$

$= L\hat{I}\frac{d}{dt}(e^{st})$

$\hat{V}\cancel{e^{st}} = sL\hat{I}\cancel{e^{st}}$

$$\boxed{\frac{\hat{V}}{\hat{I}} = sL}$$ (inductor s-impedance)





$$\hat{V}_R = \hat{I} Z_R \qquad \hat{V}_L = \hat{I} Z_L \qquad \hat{V}_C = \hat{I} Z_C$$

(Ohm's law ~~for~~ with s-impedances)

For sinusoidal:

$$u(t) = U \cos(\omega t + \phi) = U \frac{e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)}}{2}$$

$$u(t) = \underbrace{\frac{U e^{j\phi}}{2}}_{\hat{u}_1 = \tilde{u}} \cdot \underbrace{e^{j\omega t}}_{e^{s_1 t}} + \underbrace{\frac{U e^{-j\phi}}{2}}_{\tilde{u}_2 = \bar{\tilde{u}}} \underbrace{e^{-j\omega t}}_{e^{s_2 t}} \quad \text{(real)}$$

$s_1 = j\omega$ $\qquad$ $s_2 = -j\omega$

always complex-conjugates b/c $u(t)$ is real!

$$u(t) = \tilde{u}\, e^{s_1 t} + \bar{\tilde{u}}\, e^{s_2 t}$$

Use superposition to solve for the first & second component.

$$\vec{x}(t) = \vec{\tilde{x}}_1 e^{s_1 t} + \vec{\tilde{x}}_2 e^{s_2 t} \quad \text{(solution form)}$$

For $s_1 = j\omega$: $\quad M_1 = s_1 I - A(s_1) = j\omega I - A(j\omega)$

From (2)

$$M_1 \vec{\tilde{x}}_1 = \vec{\tilde{u}}$$

$$\boxed{\vec{\tilde{x}}_1 = M_1^{-1} \vec{\tilde{u}}}$$

independent sources

circuit topology

element currents & voltages $\begin{bmatrix} \vec{\tilde{I}} \\ \vec{\tilde{V}} \end{bmatrix}$

For $s_2 = \cancel{}$ $-j\omega$: $\quad M_2 = s_2 I - A(s_2) = -j\omega I - A(-j\omega)$

$$= -j\omega I - \bar{A}(j\omega) = \bar{M}_1$$

$$M_2 \vec{\tilde{x}}_2 = \vec{\tilde{u}}$$

$$\vec{\tilde{x}}_2 = M_2^{-1} \vec{\tilde{u}}$$

$$\vec{\tilde{x}}_2 = \bar{M}_1^{-1} \vec{\tilde{u}} = \overline{M_1^{-1} \vec{\tilde{u}}} = \bar{\vec{\tilde{x}}}_1$$

so $\quad \vec{x}(t) = \vec{\tilde{x}}_1 e^{s_1 t} + \vec{\tilde{x}}_2 e^{s_2 t}$ (superposition)

$$\vec{x}(t) = \vec{\tilde{x}}_1 e^{j\omega t} + \bar{\vec{\tilde{x}}}_1 e^{-j\omega t} \quad \text{(real)}$$

complex conjugates ~~so~~ always so only need to solve for $\vec{\tilde{x}}_1$.

so, all solutions :

$$\vec{V}(t) = \vec{\tilde{V}} e^{j\omega t} + \vec{\bar{\tilde{V}}} e^{-j\omega t}$$

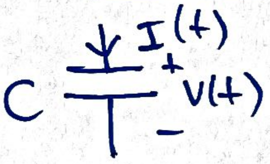
$$\vec{I}(t) = \vec{\tilde{I}} e^{j\omega t} + \vec{\bar{\tilde{I}}} e^{-j\omega t}$$

so we only need to find $(\vec{\tilde{V}}, \vec{\tilde{I}})$.

For sinusoidal inputs : $\tilde{V}, \tilde{I}$ are <u>phasors</u> (functions of $s = j\omega$)

s-impedances are called <u>impedances</u>.

<u>Phasors turn 16B problems into 16A problems !</u>



$$V(t) = V_0 \cos(\omega t + \phi)$$

$$V(t) = \underbrace{\frac{V_0}{2} e^{j\phi}}_{\tilde{V}} e^{j\omega t} + \underbrace{\frac{V_0}{2} e^{-j\phi}}_{\bar{\tilde{V}}} \cdot e^{-j\omega t}$$

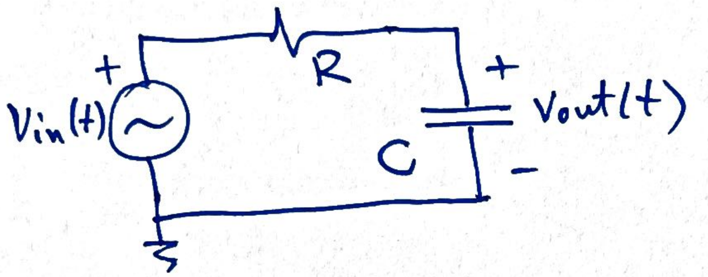
$$I(t) = C \frac{d}{dt} V(t) = C \frac{d}{dt} \left(\tilde{V} e^{j\omega t} + \bar{\tilde{V}} e^{-j\omega t} \right)$$

$$= \underbrace{j\omega C \tilde{V}}_{\tilde{I}} e^{j\omega t} + \underbrace{(-j\omega) C \bar{\tilde{V}}}_{\bar{\tilde{I}}} e^{-j\omega t}$$

$$= \tilde{I} e^{j\omega t} + \bar{\tilde{I}} e^{-j\omega t} , \quad \tilde{I} = j\omega C \tilde{V}$$

$$\frac{\tilde{V}}{\tilde{I}} = Z_C(s = j\omega) = \frac{1}{j\omega C}$$

Example 1: RC circuit



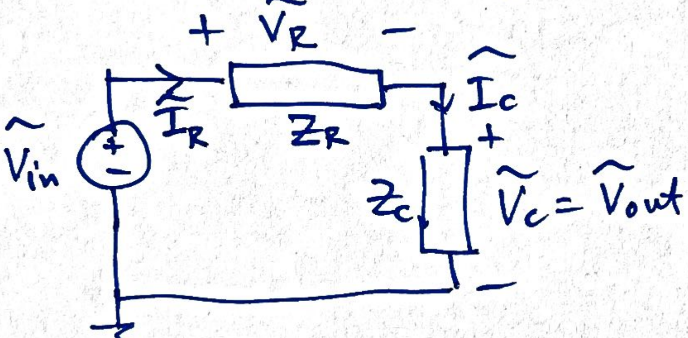
$V_{in}(t) = V_{in} \cdot \cos(\omega t + \phi)$

$V_{out}(t) = ?$

step 1: Write the independent sources as exponentials to determine the source phasors.

$$V_{in}(t) = \underbrace{\frac{V_{in}}{2} e^{j\phi}}_{\widetilde{V}_{in} \text{ phasor}} e^{+j\omega t} + \underbrace{\frac{V_{in}}{2} e^{-j\phi}}_{\widehat{V}_{in}} e^{-j\omega t}$$

step 2: Transform the circuit to phasor domain



$Z_R = R \qquad Z_c = \frac{1}{j\omega C}$

step 3: Write down the circuit equations

Elements: $\widehat{V}_R = Z_R \widehat{I}_R \qquad \widetilde{V}_c = Z_c \widehat{I}_c$

KCL: $\widehat{I}_R = \widetilde{I}_c$

Voltages: $\widehat{V}_R = \widehat{V}_{in} - \widehat{V}_c \quad , \quad \widehat{V}_c = \widehat{V}_{out}$

step 4: Solve the circuit

$$\widehat{V}_{out} = \widehat{V}_c = \frac{Z_c}{Z_c + Z_R} \widetilde{V}_{in} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} \widehat{V}_{in} = \frac{1}{1 + j\omega RC} \widehat{V}_{in}$$

$$\boxed{\tilde{V}_{out}(j\omega) = \frac{1}{1+j\omega RC} \cdot \tilde{V}_{in}(j\omega)}$$

$$\hat{V}_{in} \longrightarrow \boxed{H(j\omega)} \longrightarrow \tilde{V}_{out}$$

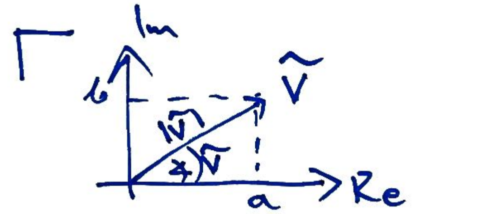
$$\boxed{H(j\omega) = \frac{\tilde{V}_{out}(j\omega)}{\hat{V}_{in}(j\omega)}}$$

transfer function

For our Low-pass RC example:

$$H_{LP}(j\omega) = \frac{1}{1+j\omega RC} = \frac{1-j\omega RC}{1+(\omega RC)^2}$$

$$\boxed{\begin{aligned} |H_{LP}(j\omega)| &= \frac{1}{|1+j\omega RC|} = \frac{1}{\sqrt{1+(\omega RC)^2}} \\ \measuredangle H_{LP}(j\omega) &= -\,\mathrm{atan2}(\omega RC, 1) \end{aligned}}$$



$\measuredangle\hat{V} = \mathrm{atan2}(b, a)$

$|\tilde{V}| = \sqrt{a^2+b^2}$

$$\tilde{V}_{out} = |\hat{V}_{out}|\, e^{j\measuredangle\hat{V}_{out}}$$

$$\hat{V}_{in} = |\hat{V}_{in}|\, e^{j\measuredangle\hat{V}_{in}}$$

$$H(j\omega) = |H(j\omega)|\, e^{j\measuredangle H(j\omega)}$$

$$\hat{V}_{out} = \underline{|\hat{V}_{out}|}\, e^{j\underline{\measuredangle\hat{V}_{out}}} = \hat{V}_{in} \cdot H(j\omega) =$$

$$= |\hat{V}_{in}|\, e^{j\measuredangle\hat{V}_{in}} \cdot |H(j\omega)|\, e^{j\measuredangle H(j\omega)}$$

$$= \underbrace{|H(j\omega)||\hat{V}_{in}|}\cdot e^{j(\underline{\measuredangle H(j\omega) + \measuredangle\hat{V}_{in}})}$$

$$|\hat{V}_{out}| = |H(j\omega)| \cdot |\hat{V}_{in}|$$

$$\measuredangle\tilde{V}_{out} = \measuredangle H(j\omega) + \measuredangle\tilde{V}_{in}$$

step 5: Convert to time domain

$$V_{out}(t) = \hat{V}_{out}\, e^{j\omega t} + \tilde{V}_{out}\, e^{-j\omega t}$$

$$= |\hat{V}_{out}|\, e^{j\measuredangle \hat{V}_{out}} \cdot e^{j\omega t} + |\hat{V}_{out}|\, e^{-j\measuredangle \hat{V}_{out}} \cdot e^{-j\omega t}$$

$$= |\hat{V}_{out}|\, e^{j(\omega t + \measuredangle \hat{V}_{out})} + |\hat{V}_{out}|\, e^{-j(\omega t + \measuredangle \hat{V}_{out})}$$

$$= 2\, |\hat{V}_{out}| \cos(\omega t + \measuredangle \hat{V}_{out})$$

similarly: $V_{in}(t) = \underbrace{2\,|\hat{V}_{in}|}_{V_{in}} \cos(\omega t + \underbrace{\measuredangle \hat{V}_{in}}_{\phi})$

$$V_{out}(t) = 2\, |H_{LP}(j\omega)|\, |\hat{V}_{in}| \cos(\omega t + \underbrace{\measuredangle \hat{V}_{in}}_{\phi} + \measuredangle H_{LP}(j\omega))$$

$$= \cancel{2}\, \frac{1}{\sqrt{1+(\omega RC)^2}} \cdot \frac{V_{in}}{\cancel{2}} \cos(\qquad)$$

$$= \frac{V_{in}}{\sqrt{1+(\omega RC)^2}} \cos(\omega t + \phi + \underbrace{\measuredangle H_{LP}(j\omega)}_{-\mathrm{atan2}(\omega RC,\, 1)})$$