Discussion 1A

1. Current, Power, and Energy for a Capacitance (Hambley Example 3.3)

Suppose that the voltage waveform shown in Figure 1 is applied to a 10-μF capacitance.

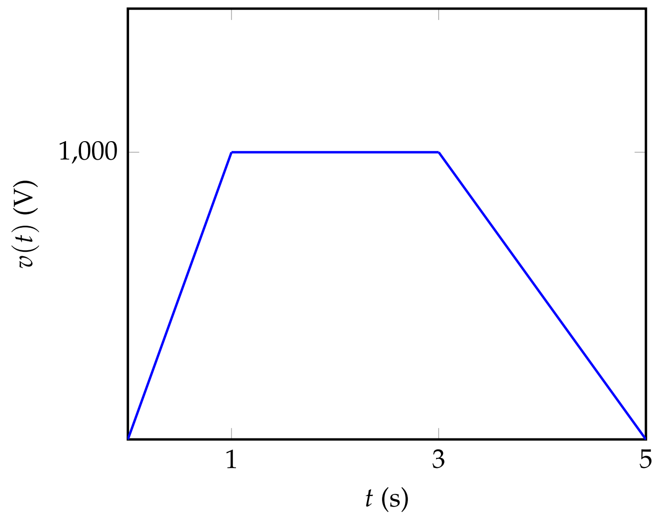

Figure 1: Plot of $v(t)$

Find and plot the current, the power delivered, and the energy stored for time between 0 and 5 s.

Solution:

First, we write expressions for the voltage as a function of time:

$$v(t) = \begin{cases} 1000t\,\text{V} & 0 < t < 1 \\ 1000\,\text{V} & 1 < t < 3 \\ 500(5-t)\,\text{V} & 3 < t < 5 \end{cases}$$

Then, using the equation

$$i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{1}$$

We can obtain expressions for the current

$$i(t) = \begin{cases} 10 \times 10^{-3}\,\text{A} & 0 < t < 1 \\ 0\,\text{A} & 1 < t < 3 \\ -5 \times 10^{-3}\,\text{A} & 3 < t < 5 \end{cases}$$

Using the equation

$$p(t) = v(t)i(t) \tag{2}$$

We can obtain expressions for power

$$p(t) = \begin{cases} 10t\,\text{W} & 0 < t < 1 \\ 0\,\text{W} & 1 < t < 3 \\ 2.5(t-5)\,\text{W} & 3 < t < 5 \end{cases}$$

Lastly, using the equation

$$E(t) = \frac{1}{2}Cv^2(t) \tag{3}$$

$$E(t) = \begin{cases} 5t^2\,\text{J} & 0 < t < 1 \\ 5\,\text{J} & 1 < t < 3 \\ 1.25(5-t)^2\,\text{J} & 3 < t < 5 \end{cases}$$

The plots for $i(t)$, $p(t)$, and $E(t)$ are as follows

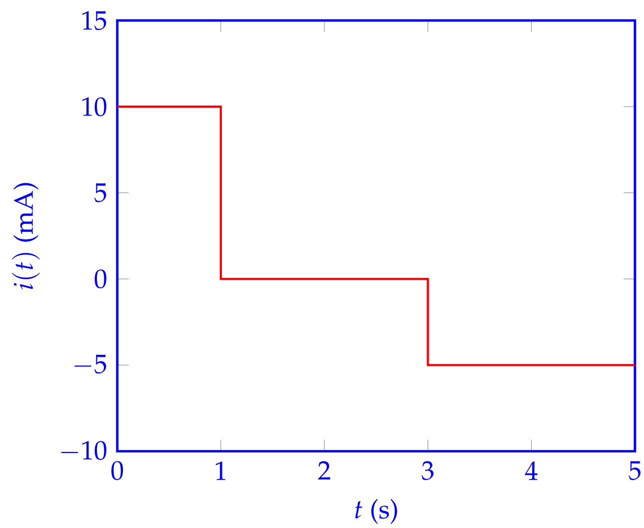

Figure 2: Plot of $i(t)$

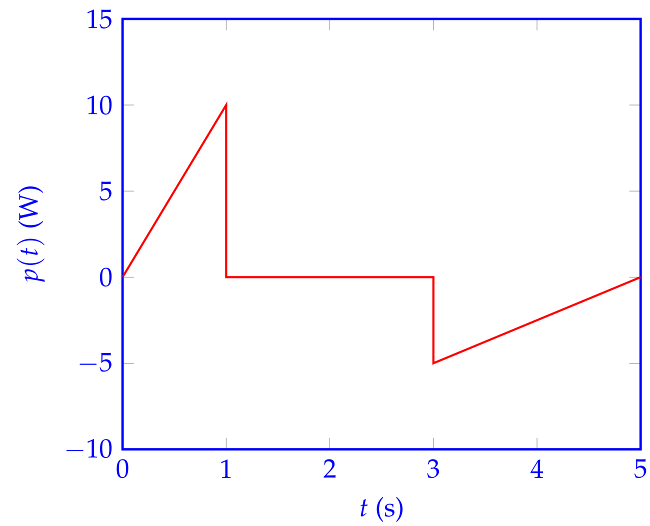

Figure 3: Plot of $p(t)$

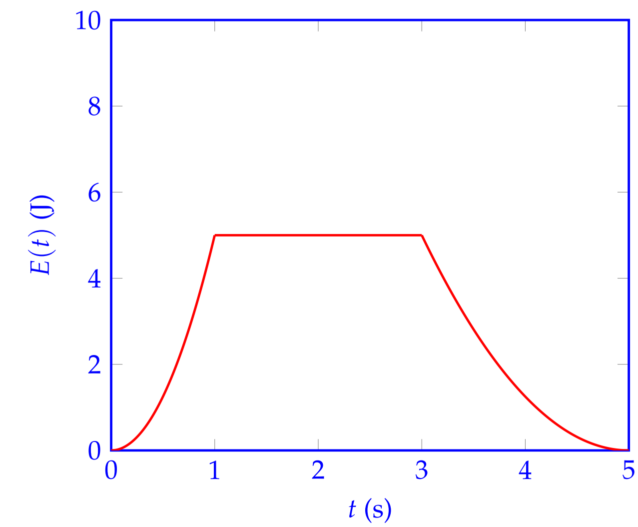

Figure 4: Plot of $E(t)$

2. Determining Voltage for a Capacitance Given Current (Hambley Example 3.2)

After t_0 the current in a 0.1 μF capacitor is given by

$$i(t) = 0.5 \sin 10^4 t \tag{4}$$

(The argument of the sin function is in radians.) The initial charge on the capacitor is $q(0) = 0$.

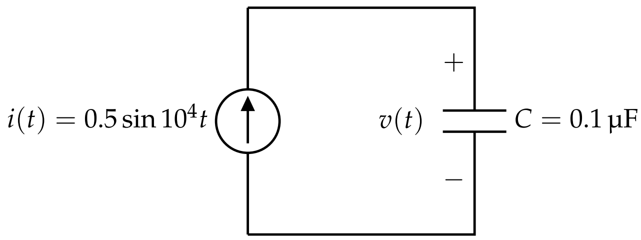

Figure 5: Example Circuit

Plot $i(t)$, $q(t)$, and $v(t)$ to scale versus time.

Solution:

Method 1 (solving for voltage first): Recall the equation relating the current and voltage of a capacitor:

$$i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{5}$$

Since we intend to solve for voltage first, we rearrange the equation accordingly and integrate:

$$\begin{aligned}
dv(t) &= \frac{1}{C} i(t) dt && (6)\\
\int_{v(0)}^{v(t)} dv(t) &= \int_0^t \frac{1}{C} i(t) dt && (7)\\
v(t) &= \frac{1}{C}\int_0^t i(t) dt + v(0) && (8)\\
&= \frac{1}{10^{-7}}\int_0^t 0.5 \sin 10^4 t dt + 0 && (9)\\
&= \frac{1}{10^{-7}} \times -0.5 \times \left[10^{-4} \cos 10^4 t\right]\Big|_0^t && (10)\\
&= \frac{1}{10^{-7}} \times 0.5 \times 10^{-4}\left[1 - \cos 10^4 t\right] && (11)\\
&= 500\left[1 - \cos 10^4 t\right] && (12)
\end{aligned}$$

The fact that $v(0)$ was zero came from the relationship $q(t) = Cv(t)$. Next, to find $q(t)$ we simply use this very relationship and multiply by capacitance.

$$q(t) = 0.5 \times 10^{-4}\left[1 - \cos 10^4 t\right] \tag{13}$$

Finally, we plot all three noted quantities (plots following method 2 solution).

Method 2 (solving for charge first): Recall the following equation

$$q(t) = \int_0^t i(t)dt + q(0) \tag{14}$$

We can directly use this equation to solve for the charge on the capacitor.

$$q(t) = \int_0^t i(t)dt + q(0) \tag{15}$$

$$= \int_0^t 0.5 \sin 10^4 t dt \tag{16}$$

$$= -0.5 \times \left[10^{-4} \cos 10^4 t\right]\Big|_0^t \tag{17}$$

$$= 0.5 \times 10^{-4}\left[1 - \cos 10^4 t\right] \tag{18}$$

Using the relationship between voltage and charge of a capacitor, we can solve for $v(t)$

$$v(t) = \frac{q(t)}{C} = \frac{q(t)}{10^{-7}} = 500\left[1 - \cos 10^4 t\right] \tag{19}$$

Here are the plots of $i(t)$, $q(t)$, and $v(t)$

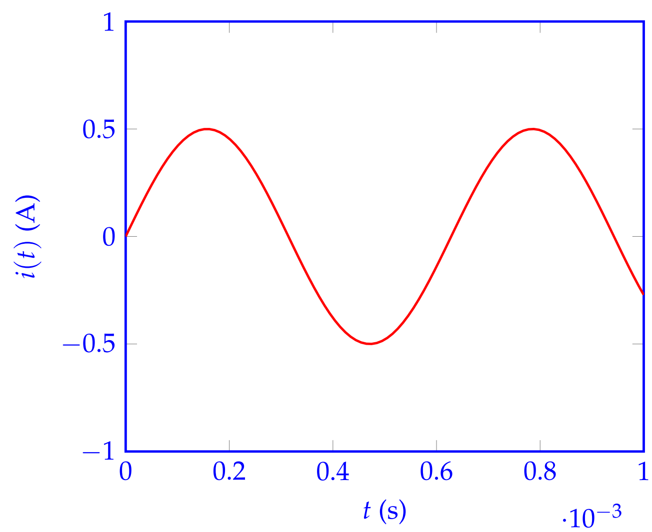

Figure 6: Plot of $i(t)$

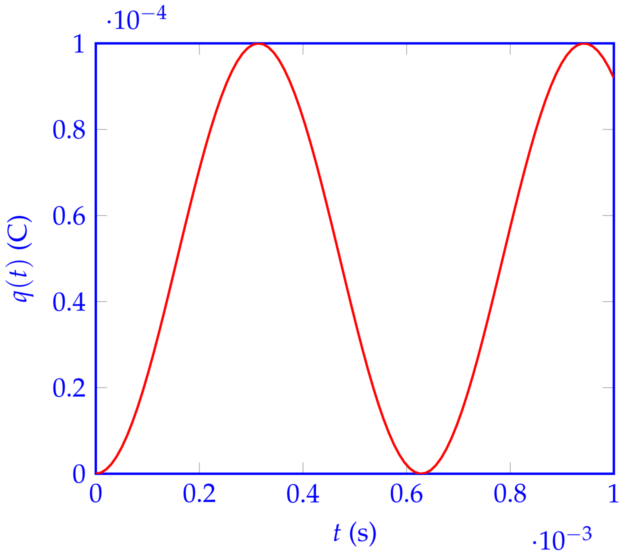

Figure 7: Plot of $q(t)$

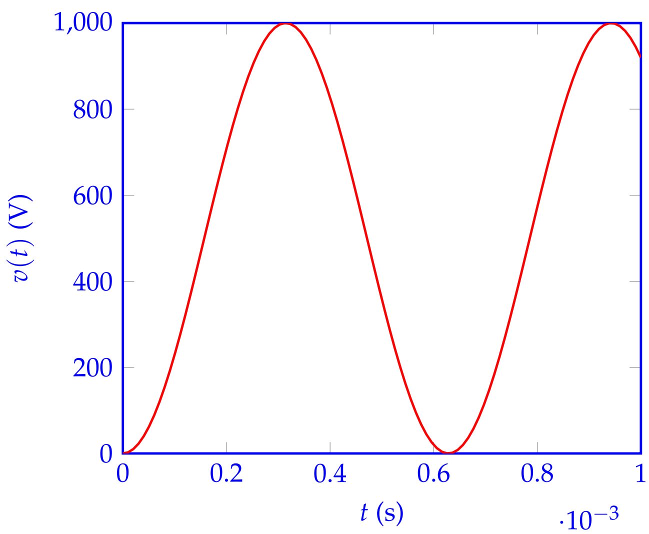

Figure 8: Plot of $v(t)$

3. Analyzing an RC Circuit with a Sinusoidal Source (Adapted from Hambley Example 4.4)

Assume you are given the following circuit, where the capacitor is initially charged so that $v_C(t) = 1V$.

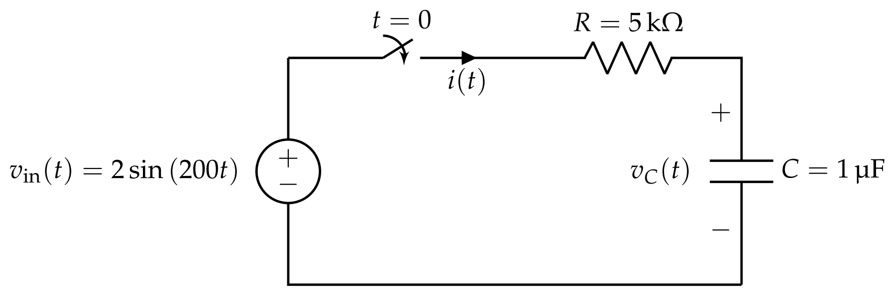

Figure 9

(a) **Set up a differential equation for the voltage $v_C(t)$ across the capacitor in the form:**

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} + av(t) = b(t) \tag{20}$$

Solution: Let's start by writing the current equation for $t > 0$ using KCL on the node that connects the resistor to the capacitor:

$$i_R(t) = i_C(t) \tag{21}$$

$$\frac{v_{in}(t) - v_C(t)}{R} = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} \tag{22}$$

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + \frac{1}{RC}v_C(t) = \frac{v_{in}(t)}{RC} \tag{23}$$

where $a = \frac{1}{RC} = \frac{1}{5\times10^{-3}} = 200$ and $b(t) = \frac{v_{in}(t)}{RC} = 400\sin(200t)$

(b) **What is the initial condition of $v(t)$? In other words, what is $v(0)$?**

Solution: We are told in the problem that $v_C(0) = 1\,\mathrm{V}$. If we were solving for current or using a more complicated setup, we may have to consider all of the components' steady-state properties.

(c) **Solve for the voltage $v(t)$ through the circuit. Also, identify the transient response (homogeneous solution) and the forced response (particular solution) of $v(t)$.** You may directly use the fact that the solution to a differential equation in the same form as Equation 20 is:

$$v(t) = Ae^{-at} + e^{-at}\int_0^t e^{at'}b(t')\,\mathrm{d}t' \tag{24}$$

(HINT: The following integral might be useful:

$$\int e^{at}\sin(bt) = \frac{1}{b^2 + a^2}e^{at}(b\sin(bt) - a\cos(bt)) \tag{25}$$

)

Solution: Now that we have our differential equation in the standard form, we can use the general form of the solution to a first-order differential equation to solve for $i(t)$. Recall that the solution is:

$$y(t) = Ae^{-at} + e^{-at}\int e^{at'}b(t')\,\mathrm{d}t' \tag{26}$$

so using out differential equation and plugging in a and $b(t)$, we get:

$$v_C(t) = A_1 e^{-\frac{1}{RC}t} + e^{-\frac{1}{RC}t}\int_0^t e^{\frac{1}{RC}t'} 400\sin(200t')\,dt' \tag{27}$$
$$v_C(t) = A_1 e^{-200t} + 400e^{-\frac{1}{RC}t}\int_0^t e^{\frac{1}{RC}t'} \sin(200t')\,dt' \tag{28}$$

From here, we can apply Equation 25 from the hint and substituting values for $R = 5\,\text{k}\Omega$ and $C = 1\,\mu\text{F}$:

$$v_C(t) = A_1 e^{-200t} + 400e^{-\frac{1}{RC}t}\left[\frac{1}{\frac{1}{RC}^2 + 200^2} e^{\frac{1}{RC}t'}\left(200\sin(200t') - \frac{1}{RC}\cos(200t')\right)\right]_0^t \tag{29}$$
$$v_C(t) = A_1 e^{-200t} + e^{-200t}\frac{400}{200^2+200^2}\left[e^{200t'}\left(200\sin(200t') - 200\cos(200t')\right)\right]_0^t \tag{30}$$
$$v_C(t) = A_1 e^{-200t} + e^{-200t}\frac{400}{200^2+200^2}\left[\left(e^{200t}\left(200\sin(200t) - 200\cos(200t)\right)\right) - (-200)\right] \tag{31}$$
$$v_C(t) = A_1 e^{-200t} + e^{-200t}\frac{400}{200^2+200^2}e^{200t}\left(200\sin(200t) - 200\cos(200t)\right) + \frac{400\cdot 200}{200^2+200^2}e^{-200t} \tag{32}$$
$$v_C(t) = \left(A_1 + \frac{400\cdot 200}{200^2+200^2}\right)e^{-200t} + \left(\sin(200t) - \cos(200t)\right) \tag{33}$$
$$v_C(t) = Ae^{-200t} + \left(\sin(200t) - \cos(200t)\right)\text{V} \tag{34}$$

Note that the extra term obtained using the lower limit of the integral was joined with the existing arbitrary constant to create another arbitrary constant and thus does not impact the solution to the problem.

Lastly, we need to solve for the value of the constant A which we will use the initial condition for.

$$v_C(0) = Ae^{-200\times 0} + \left(\sin(200\times 0) - \cos(200\times 0)\right) \tag{35}$$
$$A = 2 \tag{36}$$

Putting this all together, we get that our final solution for current $i(t)$ is:

$$v_C(t) = 2e^{-200t} + \left(\sin(200t) - \cos(200t)\right)\text{V} \tag{37}$$

where the transient response is $2e^{-200t}$ (goes to 0 over time) and the forced response is $\sin(200t) - \cos(200t)$.

(d) **(OPTIONAL) Solve for the current $i(t)$ through the circuit.**

Solution: There are two ways to go about solving for $i_C(t)$:

(1) You can solve for the voltage by either setting up a differential equation for $i_C(t)$ and solving the differential equation or

(2) use the solution from part (a) and the IV relationship for a capacitor. For this problem, we are simply going to use our solution from part (a) and plug it into the voltage of a capacitor in terms of current:

$$i_C(t) = C\frac{dv_C}{dt} \tag{38}$$

$$i_C(t) = C\frac{\mathrm{d}(2\mathrm{e}^{-200t} + (\sin(200t) - \cos(200t)))}{\mathrm{d}t} \tag{39}$$

$$i_C(t) = -400\mathrm{e}^{-200t} + (200\sin(200t) + 200\cos(200t))\,\mu\mathrm{A} \tag{40}$$