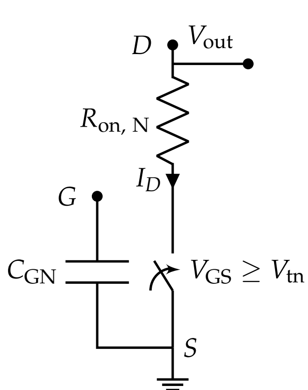
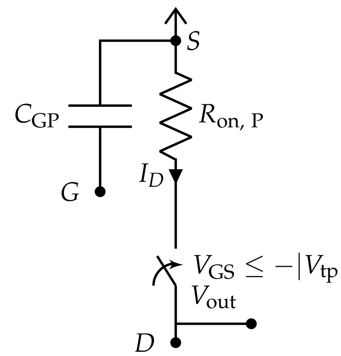
Discussion 1B

1. Transistor Switch Model

We can improve our resistor-switch model of the transistor by adding in a gate capacitance. In this model, the gate capacitances C_{GN} and C_{GP} represent the lumped physical capacitance present on the gate node of all transistor devices. This capacitance is important as it determines the delay of a transistor logic chain.



(a) NMOS Transistor Resistor-switch-capacitor model



(b) PMOS Transistor Resistor-switch-capacitor model. Note we have drawn this so that it aligns with the inverter.

You have two CMOS inverters made from NMOS and PMOS devices. Both NMOS and PMOS devices have an "on resistance" of $R_{\text{on, N}} = R_{\text{on, P}} = 1\,\text{k}\Omega$, and each has a gate capacitance (input capacitance) of $C_{\text{GN}} = C_{\text{GP}} = 1$ fF (fF = femto-Farads = 1×10^{-15} F). We assume the "off resistance" (the resistance when the transistor is off) is infinite (i.e., the transistor acts as an open circuit when off). The supply voltage V_{DD} is 1V. Assume $V_{\text{DD}} > V_{\text{tn}}, |V_{\text{tp}}| > 0$. The two inverters are connected in series, with the output of the first inverter driving the input of the second inverter (Figure 2).

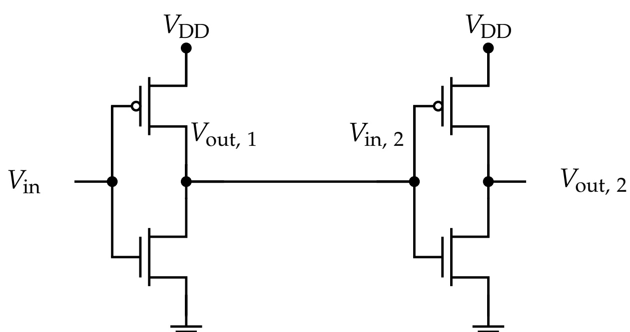


Figure 2: CMOS Inverter chain

(a) Assume the input to the first inverter has been low ($V_{\text{in}} = 0$ V) for a long time, and then switches at time $t = 0$ to high ($V_{\text{in}} = V_{\text{DD}}$).

Draw a simple RC circuit and write a differential equation describing the output voltage of the first inverter ($V_{\text{out, 1}}$) for time $t \geq 0$.

Don't forget that the second inverter is "loading" the output of the first inverter — you need to think about both of them.

(HINT: Your simple RC circuit model will only have 3 elements; you only need to draw the elements that impact the behavior of $V_{\text{out, 1}}$ and thus are relevant in this specific scenario. Also, for the first inverter, when $V_{\text{in}} = V_{\text{DD}}$, the NMOS transistor model's switch will be closed while the PMOS transistor model's switch will be open.)

Solution: To analyze this circuit as an RC circuit we can recall the transistor switch model. Using this we can see that the first inverter's output appears as a resistor connected to ground when the input turns high ($V_{\text{in}} = V_{\text{DD}}$) since only the switch for the NMOS transistor is closed.

The second inverter "loads" the output of the first inverter. From the notes in the problem, we can model the gates of the transistors as capacitors. These gates together form our capacitive load. The gate of the PMOS acts as a capacitor to V_{DD} and the gate of the NMOS acts as a capacitor to ground.

Using this we can draw the following RC circuit:

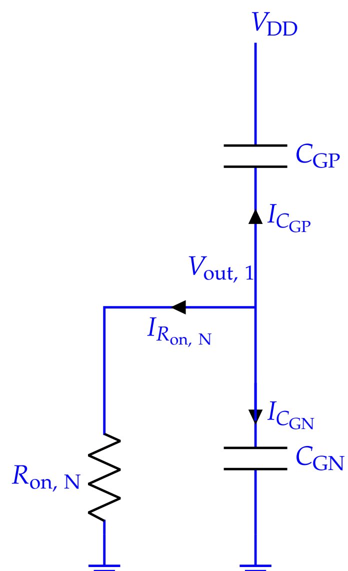


Figure 3: First inverter output at 0

We know the voltage across C_{GP} is $V_{\text{out, 1}}(t) - V_{\text{DD}}$ and the voltage across C_{GN} is $V_{\text{out, 1}}(t)$. Using this information we can set up a differential equation to solve for $V_{\text{out}}(t)$.

Writing the expressions for the three branch currents yields:

$$I_{C_{\text{GP}}} = C_{\text{GP}} \frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{out, 1}}(t) - V_{\text{DD}}) \tag{1}$$

$$I_{C_{\text{GN}}} = C_{\text{GN}} \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out, 1}}(t) \tag{2}$$

$$I_{R_{\text{on, N}}} = \frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}} \tag{3}$$

Writing KCL at the single node yields:

$$I_{C_{\text{GP}}} + I_{C_{\text{GN}}} + I_{R_{\text{on, N}}} = 0 \tag{4}$$

in other words:

$$I_{C_{GP}} + I_{C_{GN}} = -I_{R_{\text{on, N}}} \tag{5}$$

Expanding the branch currents with their expressions:

$$C_{GP}\frac{d}{dt}(V_{\text{out, 1}}(t) - V_{DD}) + C_{GN}\frac{d}{dt}V_{\text{out, 1}}(t) = -\frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}} \tag{6}$$

$$C_{GP}\frac{d}{dt}V_{\text{out, 1}}(t) + C_{GN}\frac{d}{dt}V_{\text{out, 1}}(t) = -\frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}} \tag{7}$$

$$(C_{GP} + C_{GN})\frac{d}{dt}V_{\text{out, 1}}(t) = -\frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}} \tag{8}$$

Re-writing as a first-order differential equation for $V_{\text{out, 1}}$ yields:

$$\frac{d}{dt}V_{\text{out, 1}}(t) + \frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}(C_{GP} + C_{GN})} = 0 \tag{9}$$

(b) **Solve for $V_{\text{out, 1}}(t)$.** The initial condition will be $V_{\text{out,1}}(0) = V_{DD}$ (this can be found by using the situation described in part (a)).

Solution: From our differential equation, we can notice that it is in the form

$$\frac{d}{dt}V_{\text{out, 1}}(t) + \frac{1}{\tau}V_{\text{out, 1}}(t) = 0 \tag{10}$$

where $\tau = R_{\text{on, N}}(C_{GP} + C_{GN})$.

From lecture, you may recognize that this equation is essentially the same as the differential equation for an RC circuit without inputs and the corresponding solution is simply the homogeneous solution for this differential equation

$$V_{\text{out, 1}}(t) = Ke^{-\frac{t}{\tau}} = Ke^{-\frac{t}{R_{\text{on, N}}(C_{GP}+C_{GN})}} \tag{11}$$

We can then use our initial condition:

$$V_{\text{out, 1}}(0) = Ke^{-\frac{0}{R_{\text{on, N}}(C_{GP}+C_{GN})}} \tag{12}$$

$$V_{DD} = K \tag{13}$$

Thus, our final solution is

$$V_{\text{out, 1}}(t) = V_{DD}e^{-\frac{t}{R_{\text{on, N}}(C_{GP}+C_{GN})}} \tag{14}$$

(c) **Sketch the output voltage of the first inverter, showing clearly (1) the initial value, (2) the asymptotic value, and (3) the time that it takes for the voltage to decay to roughly 1/3 of its initial value.** *(HINT: For part (3), use the approximation that* $e^{-1} = \frac{1}{e} \approx \frac{1}{3}$.*)*

Solution:

(1) We know that the output of our inverter started with the initial value V_{DD}.

(2) We can find the asymptotic value by plugging in $t = \infty$ to the solution we found for $V_{\text{out, 1}}(t)$ to find $V_{\text{out, 1}} = V_{\text{DD}} e^{-\frac{\infty}{R_{\text{on, N}}(C_{\text{GP}}+C_{\text{GN}})}} = 0$.

(3) To approximate when the output will decay to $\frac{1}{3}$ its original value, we use the fact that $e^{-1} = \frac{1}{e} \approx \frac{1}{3}$. We thus want to find when $V_{\text{out, 1}} = V_{\text{DD}} e^{-1}$.
This will occur when the e term is raised to -1, which occurs when $t = \tau = R_{\text{on, N}}(C_{\text{GP}} + C_{\text{GN}}) = 2 \times 10^{-12}$seconds.
Note the significance of the time constant τ; as defined in the differential equation and solution to the differential equation, it provides a measure of how much time it takes for a system to reach its steady state, which can be compared between different systems to compare their speeds.

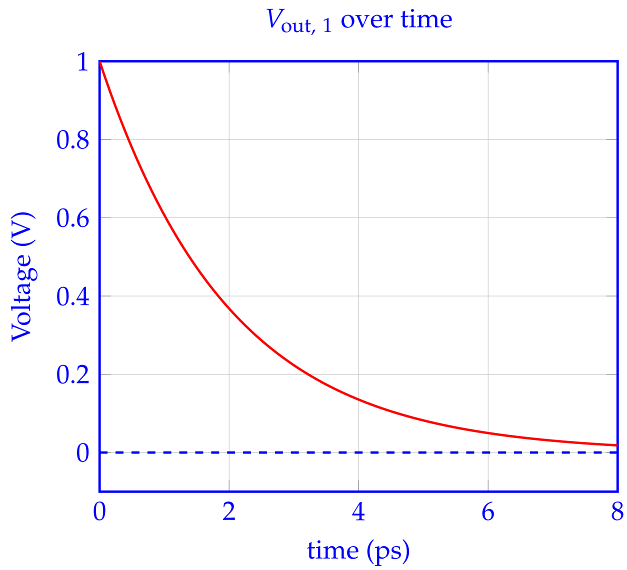


Figure 4

2. **Complex Algebra (Review)**

(a) **Express the following values in polar forms:** -1, j, $-$j, $(\mathrm{j})^{\frac{1}{2}}$, **and** $(-\mathrm{j})^{\frac{1}{2}}$. Recall $\mathrm{j}^2 = -1$, and the complex conjugate is defined as follows: for a complex number $z = x + \mathrm{j}y$, the complex conjugate $\overline{z} = x - \mathrm{j}y$. (Note: The complex conjugate is also sometimes notated as $z^* = x - \mathrm{j}y$, which is equivalent to $\overline{z} = x - \mathrm{j}y$.)

Solution: Here, we review some basic properties of complex numbers and its rectangular and polar form: $z = x + \mathrm{j}y = |z|\mathrm{e}^{\mathrm{j}\theta}$, where $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$ and $\measuredangle z = \theta = \operatorname{atan2}(y, x)$. We can also write $x = |z|\cos(\theta)$, $y = |z|\sin(\theta)$.

A complex number can be represented in the following forms:

$$z = a + \mathrm{j}b = r\cos(\theta) + \mathrm{j}r\sin(\theta) = r\mathrm{e}^{\mathrm{j}\theta}, \tag{15}$$

where, $r = \sqrt{a^2 + b^2}$,$\measuredangle z = \operatorname{atan2}(b, a)$ and a, b are real numbers.

$$-1 = \mathrm{j}^2 = \mathrm{e}^{\mathrm{j}\pi} = \mathrm{e}^{-\mathrm{j}\pi} \tag{16}$$

$$\mathrm{j} = \mathrm{e}^{\mathrm{j}\frac{\pi}{2}} = \sqrt{-1} \tag{17}$$

$$-\mathrm{j} = -\mathrm{e}^{\mathrm{j}\frac{\pi}{2}} = \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}} \tag{18}$$

$$(\mathrm{j})^{\frac{1}{2}} = (\mathrm{e}^{\mathrm{j}\frac{\pi}{2}})^{\frac{1}{2}} = \mathrm{e}^{\mathrm{j}\frac{\pi}{4}} = \frac{1 + \mathrm{j}}{\sqrt{2}} \tag{19}$$

$$(-\mathrm{j})^{\frac{1}{2}} = (\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}})^{\frac{1}{2}} = \mathrm{e}^{-\mathrm{j}\frac{\pi}{4}} = \frac{1 - \mathrm{j}}{\sqrt{2}} \tag{20}$$

(b) Represent $\sin(\omega t + \theta)$ and $\cos(\omega t + \theta)$ using complex exponentials. (*Hint:* Use Euler's identity $\mathrm{e}^{\mathrm{j}\phi} = \cos(\phi) + \mathrm{j}\sin(\phi)$.)

Solution: Note that we can use the fact that $\cos(x)$ is an even function, and $\sin(x)$ is an odd function. This gives us that:

$$\begin{aligned}
\mathrm{e}^{\mathrm{j}(\omega t + \theta)} &= \cos(\omega t + \theta) + \mathrm{j}\sin(\omega t + \theta) \\
\mathrm{e}^{-\mathrm{j}(\omega t + \theta)} &= \cos(-(\omega t + \theta)) + \mathrm{j}\sin(-(\omega t + \theta)) \\
&= \cos(\omega t + \theta) - \mathrm{j}\sin(\omega t + \theta)
\end{aligned}$$

Solving this system of equations for $\cos(\omega t + \theta)$ and $\sin(\omega t + \theta)$ gives:

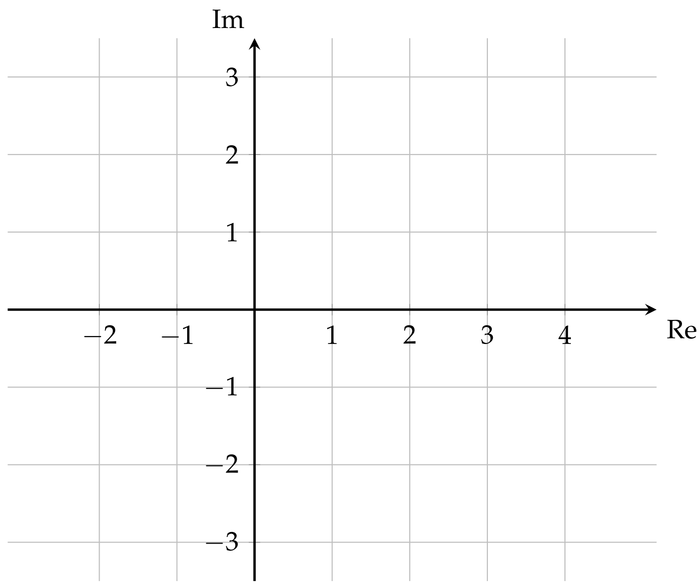
$$\sin(\omega t + \theta) = \frac{\mathrm{e}^{\mathrm{j}(\omega t + \theta)} - \mathrm{e}^{-\mathrm{j}(\omega t + \theta)}}{2\mathrm{j}} \qquad \cos(\omega t + \theta) = \frac{\mathrm{e}^{\mathrm{j}(\omega t + \theta)} + \mathrm{e}^{-\mathrm{j}(\omega t + \theta)}}{2}$$

Using properties of exponentials, we can also separate the exponents:

$$\sin(\omega t + \theta) = \frac{\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{\mathrm{j}\omega t} - \mathrm{e}^{-\mathrm{j}\theta}\mathrm{e}^{-\mathrm{j}\omega t}}{2\mathrm{j}} \qquad \cos(\omega t + \theta) = \frac{\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{\mathrm{j}\omega t} + \mathrm{e}^{-\mathrm{j}\theta}\mathrm{e}^{-\mathrm{j}\omega t}}{2}$$

For the next parts, let $a = 1 - \mathrm{j}\sqrt{3}$ and $b = \sqrt{3} + \mathrm{j}$.

(c) Show the number a in complex plane, marking the distance from origin and angle with real axis.



Solution: The location of a in the complex plane is shown in Figure 5. The only two pieces of information we need are the magnitude and the phase, which is the polar coordinates interpretation. We could also use the (perhaps more familiar) x and y Cartesian coordinates.

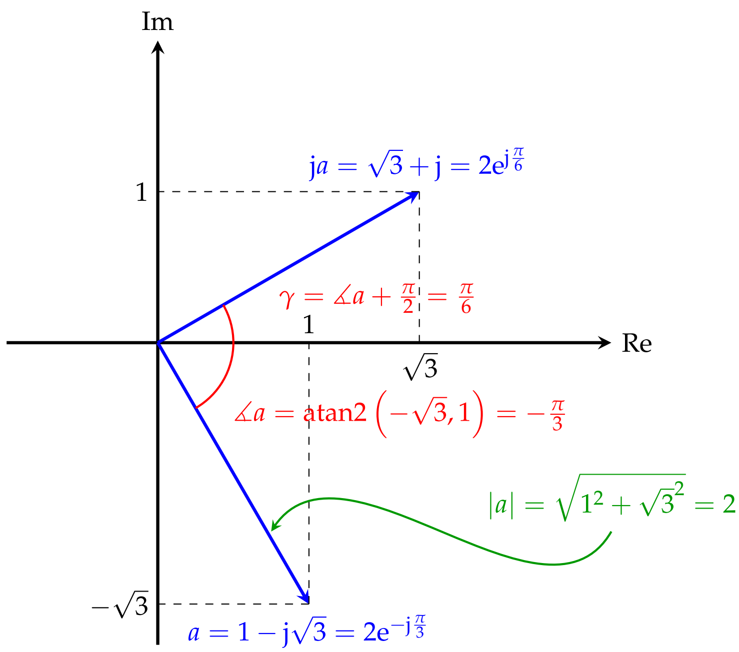


Figure 5: Complex numbers a and its rotated version b represented as vectors in the complex plane.

(d) Show that multiplying a with j is equivalent to rotating the complex number by $\frac{\pi}{2}$ or 90° in the complex plane.

Solution: Multiplying a by j:

$$ja = e^{j\pi/2} \cdot 2e^{-j\pi/3} = 2e^{j\pi/6} = \sqrt{3} + j$$

The rotation is demonstrated in the same complex plane plot (Figure 5), with a new angle $\gamma = \measuredangle a + \frac{\pi}{2}$.

(e) **(Practice)** For complex number $z = x + jy$ show that $|z| = \sqrt{z\overline{z}}$, where $\overline{z}$ is the complex conjugate of z.

Solution: We can follow the definition of complex conjugate and magnitude:

$$\sqrt{z\overline{z}} = \sqrt{(x + jy)(x - jy)} = \sqrt{x^2 + y^2} = |z| \tag{21}$$

(f) **(Practice)** Express a and b in polar form.

Solution: Following the definitions in part a):

$$|a| = 2$$
$$|b| = 2$$
$$\measuredangle a = -\frac{\pi}{3}$$
$$\measuredangle b = \frac{\pi}{6}$$

Hence:

$$a = 2e^{-j\frac{\pi}{3}} \qquad b = 2e^{j\frac{\pi}{6}}$$

(g) **(Practice)** Find ab, $a\overline{b}$, $\frac{a}{b}$, $a + \overline{a}$, $a - \overline{a}$, $\overline{ab}$, $\overline{a}\overline{b}$, and $(b)^{\frac{1}{2}}$.

Solution: We can evaluate these sequentially using the rules of complex algebra:

$$ab = 4 \cdot e^{-j\frac{\pi}{6}} = 2\sqrt{3} - 2j$$
$$a\overline{b} = 4 \cdot e^{-j\frac{\pi}{2}} = -4j$$
$$\frac{a}{b} = e^{-j\frac{\pi}{2}} = -j$$
$$a + \overline{a} = 2$$
$$a - \overline{a} = -2j\sqrt{3}$$
$$\overline{ab} = 2\sqrt{3} + 2j$$
$$\overline{a}\overline{b} = (1 + j\sqrt{3})(\sqrt{3} - j) = \sqrt{3} + \sqrt{3} + j(3 - 1) = 2\sqrt{3} + 2j$$
$$(b)^{\frac{1}{2}} = \sqrt{2}e^{j\frac{\pi}{12}}$$

Note the following: $a + \overline{a}$ is a purely real number. $a - \overline{a}$ is a purely imaginary number. And, $\overline{ab} = \overline{a}\overline{b}$.

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