Discussion 2B

1. Introduction to Inductors

An inductor is a circuit element analogous to a capacitor; its voltage is proportional to the derivative of the current across it. That is:

$$V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} \tag{1}$$

When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 1, we form the counterpart circuit for an inductor:

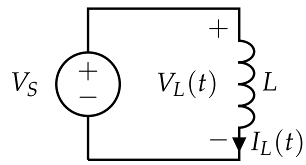


Figure 1: Inductor in series with a voltage source.

(a) **What is the current through an inductor as a function of time? If the inductance is $L = 3\,\mathrm{H}$, what is the current at $t = 6\,\mathrm{s}$?** Assume that the voltage source turns from $0\,\mathrm{V}$ to $5\,\mathrm{V}$ at time $t = 0\,\mathrm{s}$, and there's no current flowing in the circuit before the voltage source turns on, i.e $I_L(0) = 0\,\mathrm{A}$.

Solution: We proceed to analyze the given equation. Note that the voltage source is held at a constant value for $t \geq 0$, which allows us to express the derivative of current as a constant:

$$V_L(t) = L\frac{\mathrm{d}I_L}{\mathrm{d}t}$$
$$\frac{V_S}{L} = \frac{\mathrm{d}I_L}{\mathrm{d}t}$$

From here, we can see that the derivative of the current is a constant with respect to time! This immediately indicates that we have a linear relationship between current and time, with a slope set by the derivative. This means that the current in the inductor is given by

$$I_L(t) = I_L(0) + \frac{V_S}{L}t \tag{2}$$

This is exactly how we came up with the equation for the voltage across a capacitor in series with a constant current source. So, the current in the inductor keeps growing over time! Inductors store energy in their magnetic field, so the more time that this voltage source feeds the inductor, the higher the current, and the greater the stored energy.

Substituting in the specific values asked for, $I_L(6\,\mathrm{s}) = \frac{5\,\mathrm{V}}{3\,\mathrm{H}} \cdot 6\,\mathrm{s} = 10\,\mathrm{A}$.

(b) Now, we add some resistance in series with the inductor, as in Figure 3.

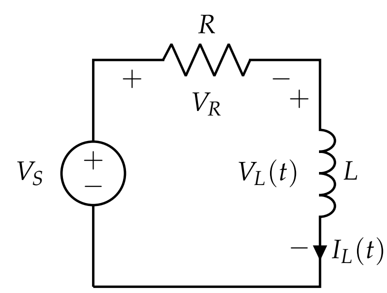


Figure 2: Inductor in series with a voltage source.

Solve for the current $I_L(t)$ and voltage $V_L(t)$ in the circuit over time, in terms of R, L, V_S, t. Note that $I_L(0) = 0\,\text{A}$. Try to solve this equation by inspection. Otherwise, you can use the following integral for the particular solution (with the proper values and functions):

$$e^{-st}\int e^{st}b(t)dt$$

Solution: We begin by considering the voltage drop across the resistor, in terms of source voltage and inductor voltage. There's also only a single current in the circuit (the one we're solving for, $I(t)$):

$$\begin{aligned} V_R(t) &= V_S - V_L(t) \\ R\,I_L(t) &= V_S - L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}I_L(t) + \frac{R}{L}I_L(t) &= \frac{V_S}{L} \end{aligned}$$

We recognize this as a first-order differential equation with an input!

Method 1: Inspection

After a long time, I_L will reach some steady state value. In steady state, an inductor behaves as a short circuit so if we replace the inductor with a short circuit, we can find the steady state current through it. In doing so, we can visualize the following circuit:

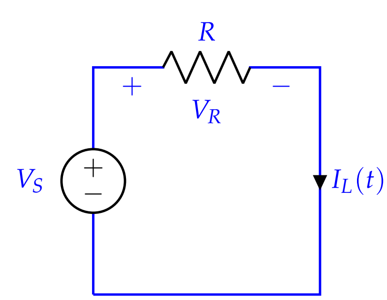


Figure 3: Inductor in series with a voltage source.

The current in this case would simply be $\lim_{t\to\infty} I_L(t) = \frac{V_S}{R}$.

From our differential equation, we can recognize that our time constant is $\tau = \frac{L}{R}$. Additionally, we know that I_L goes from $I_L(0) = 0$ to $\lim_{t\to\infty} I_L(t) = \frac{V_S}{R}$ exponentially, so the term that describes this transistion is $1 - e^{-\frac{t}{\tau}} = 1 - e^{-\frac{R}{L}t}$.

© UCB EECS 16B, Spring 2023. All Rights Reserved. This may not be publicly shared without explicit permission.

Combining our ideas, we can determine that $i_L(t) = \frac{V_S}{R}(1 - e^{-\frac{R}{L}t})$.

Method 2: Homogeneous and Particular Solution

Let's solve the differential equation by finding a homogeneous and particular solution.

Let $I_h(t)$ be a homogeneous solution. To find it, set the input term to 0 to find the relevant differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} I_h(t) + \frac{R}{L} I_h(t) = 0$$

Notice that this differential equation is similar to the RC differential equation! If we set $\tau = \frac{L}{R}$, we can find an identical solution:

$$I_h(t) = A_1 \mathrm{e}^{-\frac{t}{\tau}} = A_1 \mathrm{e}^{-\frac{R}{L}t}$$

Let $I_p(t)$ be a particular solution to our differential equation. We can find it either by using the integrating factor or directly using the provided integral (which is derived using the integrating factor). Here, we will directly use the integral (the bounds of the integral will not matter since the constant term from the integral will eventually combine with the arbitrary constant A_1 from the homogeneous solution so you could just evaluate the integral as an indefinite integral, but we will use bounds of 0 and t for formality).

For our case, $s = \frac{1}{\tau} = \frac{R}{L}$ and $b(t) = \frac{V_S}{L}$.

$$\begin{aligned} I_p(t) &= e^{-\frac{R}{L}t} \int_0^t e^{\frac{R}{L}t'} \frac{V_S}{L} dt' \\ &= e^{-\frac{R}{L}t} \left[\frac{V_S}{R} e^{\frac{R}{L}t'} \right]_0^t \\ &= e^{-\frac{R}{L}t} \left[\frac{V_S}{R} e^{\frac{R}{L}t} - \frac{V_S}{R} \right] \\ &= \frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{R}{L}t} \end{aligned}$$

Now, we can combine the two solutions to get the overall solution:

$$\begin{aligned} I_L(t) &= I_h(t) + I_p(t) \\ &= A_1 e^{-\frac{R}{L}t} + \frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{R}{L}t} \\ &= \left(A_1 - \frac{V_S}{R} \right) e^{-\frac{R}{L}t} + \frac{V_S}{R} \\ &= A e^{-\frac{R}{L}t} + \frac{V_S}{R} \end{aligned}$$

Notice that we defined $A = A_1 - \frac{V_S}{R}$, which is just another version of the arbitrary constant that will be set by the initial condition (this is why the bounds of the integral were not necessary).

Finally, we can use our initial condition ($I_L(0) = 0$) to solve for A.

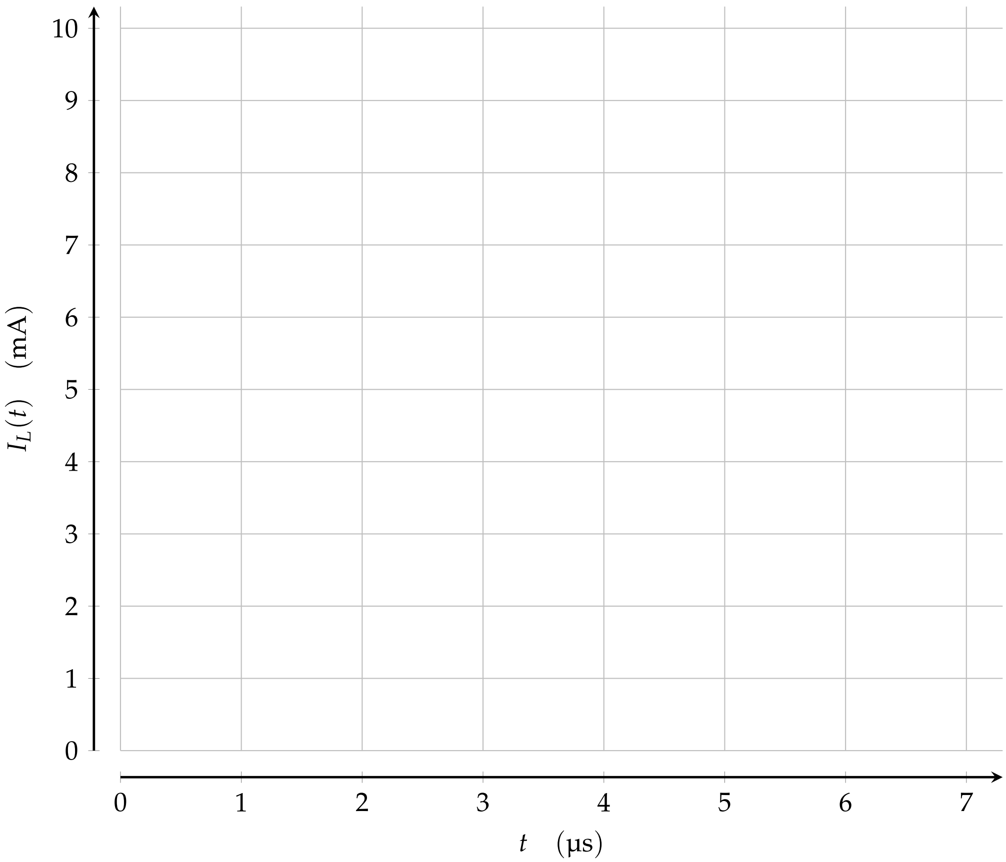
$$\begin{aligned} I_L(0) = A e^{-\frac{R}{L}(0)} + \frac{V_S}{R} &= 0 \\ A &= -\frac{V_S}{R} \end{aligned}$$

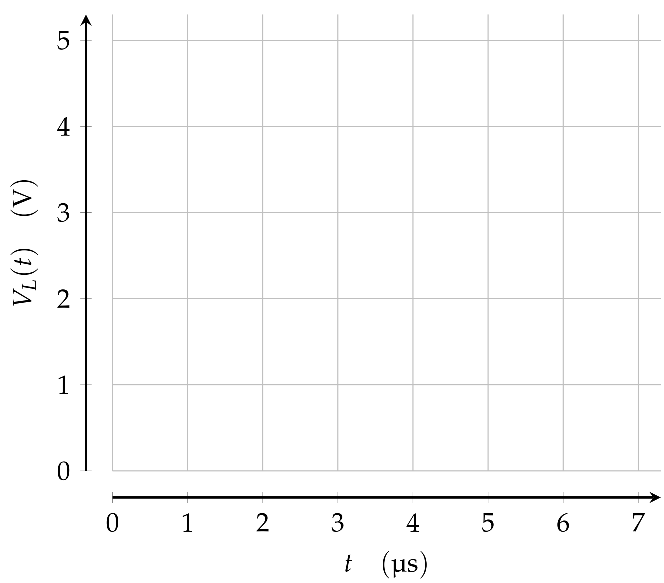
© UCB EECS 16B, Spring 2023. All Rights Reserved. This may not be publicly shared without explicit permission.

Thus, our final solution is

$$I_L(t) = -\frac{V_S}{R}e^{-\frac{R}{L}t} + \frac{V_S}{R} = \frac{V_S}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

(c) **Suppose $R = 500\,\Omega, L = 1\,\text{mH}, V_S = 5\,\text{V}$. Plot the current through and voltage across the inductor ($I_L(t), V_L(t)$), as these quantities evolve over time.**





Solution: The current begins at 0 A and over time, the inductor begins to look like a short. In the long-term, the current settles to $\frac{V_S}{R}$ A $= 1$ mA. The voltage begins at $V_S = 5$ V because the inductor initially looks like an open circuit, and this voltage decreases exponentially over time down to zero.

The time constant governing both of these transient curves is $\tau = \frac{L}{R} = 2\,\mu$s. Using this information, we can sketch the curves for current (Figure 4) and inductor voltage (Figure 5). Notice that it is perfectly fine for the voltage to be discontinuous, but the same is not true for the current.

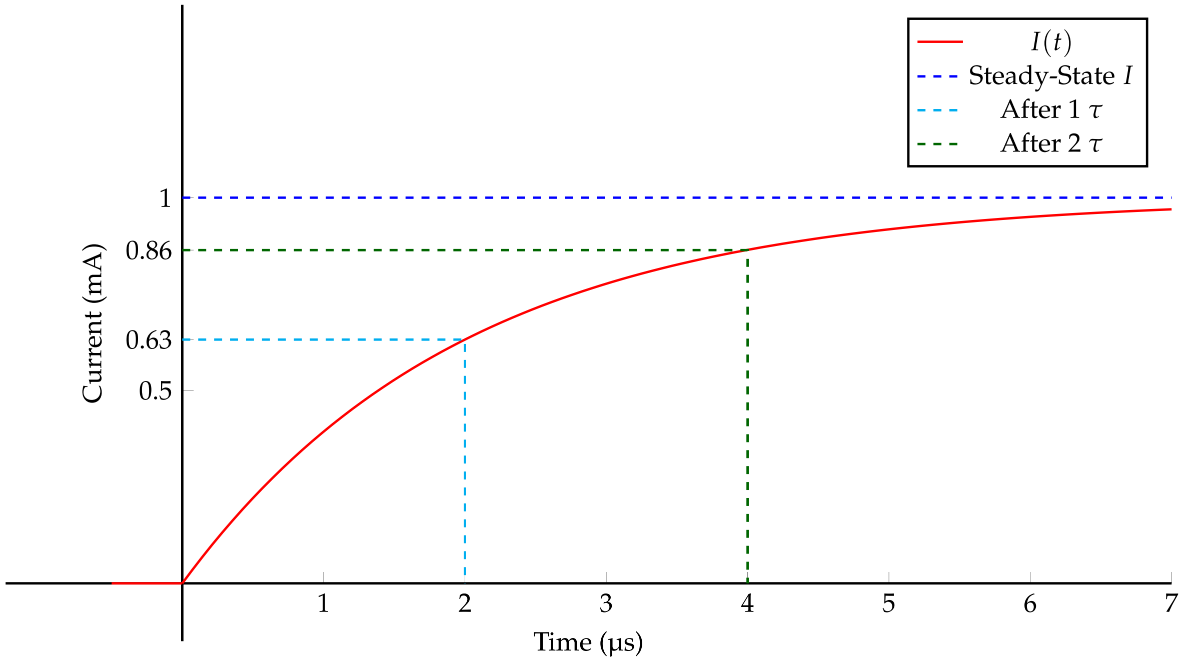


Figure 4: Transient Current in an RL circuit (with initial current $I(0) = 0$ A.)

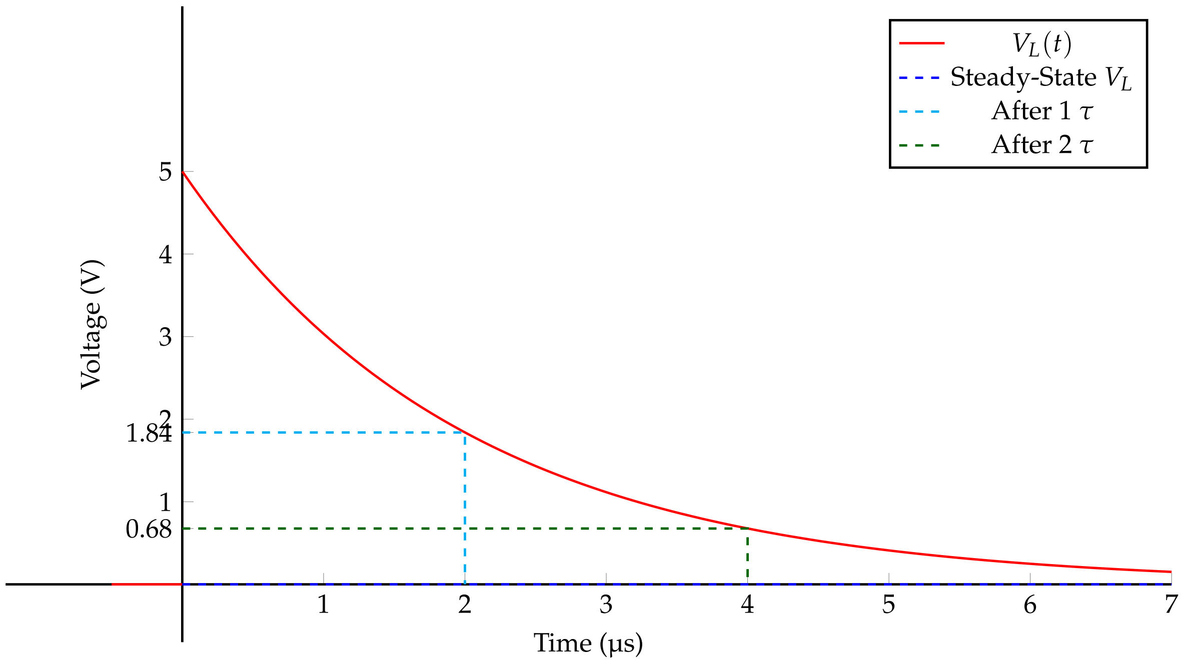


Figure 5: Transient Voltage across the inductor in an RL circuit (with initial current $I(0) = 0$ A.)

© UCB EECS 16B, Spring 2023. All Rights Reserved. This may not be publicly shared without explicit permission.

2. RL Circuit Solution Methods

Consider the following circuit:

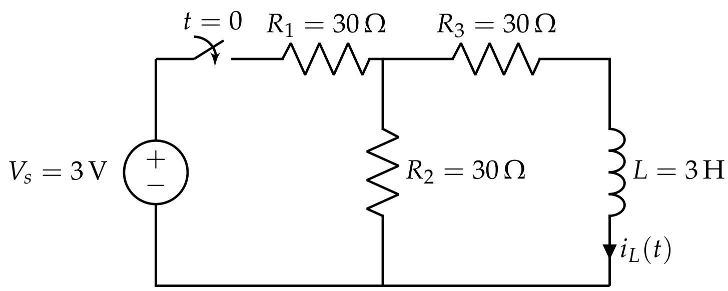


Figure 6

Before time $t = 0$, the circuit reaches a steady state. At time $t = 0$, the switch is closed. Our goal is to find the differential equation for the current through the inductor ($i_L(t)$). One method to approach this problem is to simply use Node Voltage Analysis (NVA). To start, we would define the node voltages in our circuit (including a ground node).

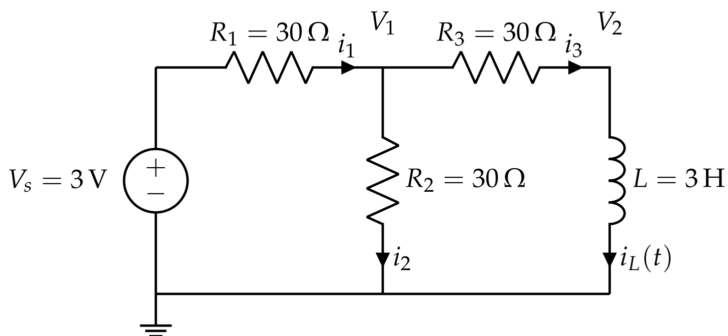


Figure 7

Then, we can set up a system of equations using KCL/KVL to find our desired differential equation.

First, let's perform KCL on the node with defined voltage V_1.

$$\begin{aligned}
i_1 &= i_2 + i_3 \\
\frac{V_s - V_1}{R_1} &= \frac{V_1 - 0}{R_2} + \frac{V_1 - V_2}{R_3} \\
\frac{3 - V_1}{30} &= \frac{V_1 - 0}{30} + \frac{V_1 - V_2}{30} \\
V_1 &= 1 + \frac{V_2}{3}
\end{aligned}$$

Now, let's perform KCL on the node with the defined voltage V_2.

Note that $V_2 - 0 = V_2$ is the voltage across the inductor so by the inductor I-V relationship, $V_2 = L\frac{\mathrm{d}i_L}{\mathrm{d}t} = 3\frac{\mathrm{d}i_L}{\mathrm{d}t}$.

$$i_3 = i_L$$

 All Rights Reserved. This may not be publicly shared without explicit permission.

$$\frac{V_1 - V_2}{R_3} = i_L$$
$$\frac{V_1 - V_2}{30} = i_L$$
$$\frac{V_1}{30} = \frac{V_2}{30} + i_L$$
$$\frac{1}{30}\left(1 + \frac{V_2}{3}\right) = \frac{V_2}{30} + i_L$$
$$\frac{1}{45}V_2 + i_L = \frac{1}{30}$$
$$\frac{1}{45}\left(3\frac{\mathrm{d}i_L}{\mathrm{d}t}\right) + i_L = \frac{1}{30}$$
$$\frac{\mathrm{d}i_L}{\mathrm{d}t} + 15i_L = \frac{1}{2}$$

Thus, we have found the differential equation! However, this method required solving a system of equations; is there another way?

(a) Another way to approach the problem is to use equivalence. Simplify the voltage source and resistor network into a voltage source and resistor using Thevenin equivalence. Then, reconnect the inductor and **find the differential equation for** $i_L(t)$.

For reference, here is the circuit that we want to simplify using Thevenin equivalence:

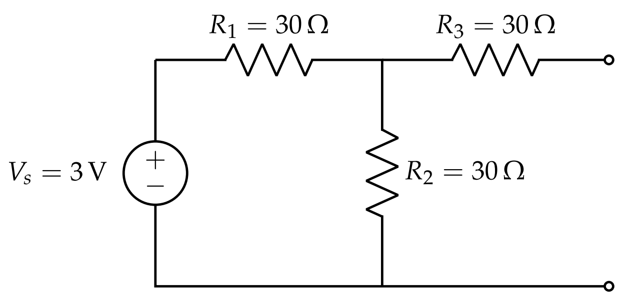


Figure 8

(HINT: Your final differential equation should be the same as the one from the problem introduction.)

Solution: There are many approaches for finding the Thevenin equivalent circuit. Let's find the voltage V_{TH} when the terminals are open and the equivalent resistance R_{TH} looking into the terminals.

To find the voltage V_{TH}, we can notice that no current flows through resistor R_3 due to the open circuit. Thus, the voltage at the terminals is the same as the voltage of the node in between all of the resistors, if we define the bottom node to be ground. Then, since the current through R_1 and R_2 must be the same by KCL, V_{TH} will just be the result of a voltage divider between those two resistors.

$$V_{TH} = \frac{R_2}{R_1 + R_2}V_s = \frac{30}{30 + 30}(3) = \frac{3}{2}$$

To find the equivalent resistance looking into the terminals, we zero out the independent voltage source (which becomes a short circuit) and find the equivalent resistance of the remaining resistors:

 All Rights Reserved. This may not be publicly shared without explicit permission.

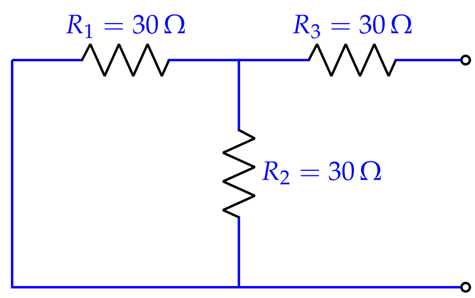


Figure 9

Using parallel/series resistance knowledge, we can find that

$$R_{TH} = R_1||R_2 + R_3 = 30||30 + 30 = 15 + 30 = 45$$

Thus, our Thevenin equivalent circuit is:

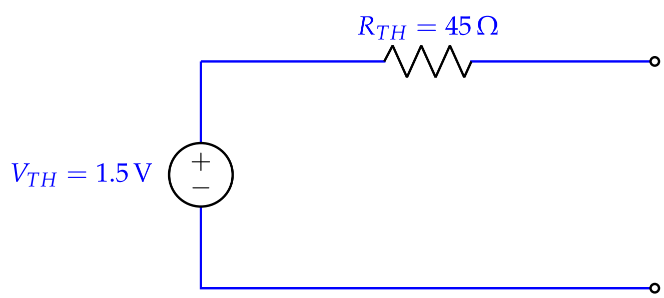


Figure 10

Now, let's add our inductor back into the circuit:

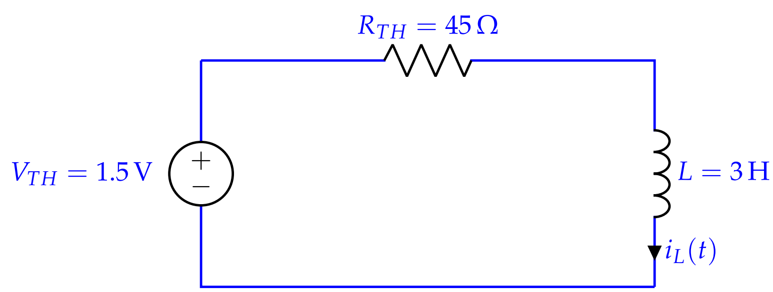


Figure 11

This is a much simpler circuit to analyze! Let's define the voltage across the inductor to be v_L and perform KCL to find the differential equation:

$$\frac{V_{TH} - v_L}{R_{TH}} = i_L$$
$$\frac{1.5 - v_L}{45} = i_L$$
$$\frac{v_L}{45} + i_L = \frac{3}{2}$$

 All Rights Reserved. This may not be publicly shared without explicit permission.

$$\frac{1}{45}\left(3\frac{\mathrm{d}i_L}{\mathrm{d}t}\right) + i_L = \frac{1}{30}$$
$$\frac{1}{15}\frac{\mathrm{d}i_L}{\mathrm{d}t} + i_L = \frac{1}{30}$$
$$\frac{\mathrm{d}i_L}{\mathrm{d}t} + 15i_L = \frac{1}{2}$$

Notice that this is the same differential equation as obtained using Node Voltage Analysis (NVA)!

(b) Now, let's start solving the differential equation. First, **find the initial condition $i_L(0)$ for our system**. Remember that the current through the inductor cannot change instantaneously (since this would correspond to infinite voltage through the inductor I-V relationship) so $i_L(0)$ will be the same as the steady state value from $t < 0$.

(HINT: If there is no voltage/current sources connected to this system, can there be any nonzero currents / voltage differences in the system during steady-state?)

Solution: Since no voltage/current sources are connected for $t < 0$ when the switch is open, the current in steady state will be $i_L(0) = 0$.

(c) **(OPTIONAL)** Now that we have our differential equation and initial condition, we can now solve for the current $i_L(t)$ as a function of time. **Solve the system for** $i_L(t)$. If you can, try to solve this by inspection. Otherwise, you can use the following integral to find the particular solution (remember to use the values/functions that correspond to this specific differential equation):

$$\mathrm{e}^{-st}\int \mathrm{e}^{st}b(t)dt$$

Solution:

Method 1: Inspection

We know that when the switch closes, the voltage source becomes connected to the system and after a long time, i_L will reach some steady state value. In steady state, an inductor behaves as a short circuit so if we replace the inductor with a short circuit, we can find the steady state current through it. In doing so, we can visualize the following circuit:

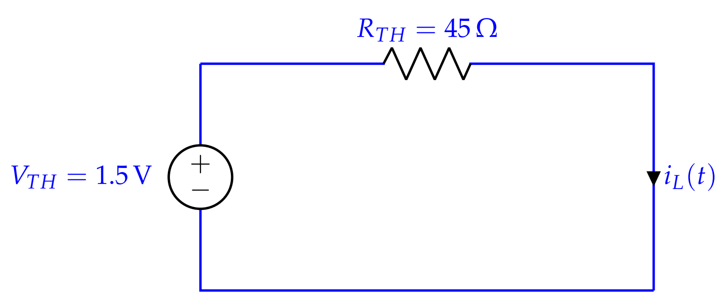


Figure 12

The current in this case would simply be $\lim_{t\to\infty} i_L(t) = \frac{V_{TH}}{R_{TH}} = \frac{1.5}{45} = \frac{1}{30}$.

From our differential equation, we can recognize that our time constant is $\tau = \frac{L}{R_{TH}} = \frac{1}{15}$. Additionally, we know that i_L goes from $i_L(0) = 0$ to $\lim_{t\to\infty} i_L(t) = \frac{1}{30}$ exponentially, so the term that describes this transistion is $1 - e^{-\frac{t}{\tau}} = 1 - e^{-15t}$.

Combining our ideas, we can determine that $i_L(t) = \frac{1}{30}(1 - e^{-15t})$.

© UCB EECS 16B, Spring 2023. All Rights Reserved. This may not be publicly shared without explicit permission.

Method 2: Homogeneous and Particular Solutions

Notice that our differential equation has an input term (not homogeneous). Thus, we will need to find both a homogeneous solution and particular solution.

Let $i_h(t)$ be a homogeneous solution to our equation. To find $i_h(t)$, set the input term in our differential equation to 0:

$$\frac{\mathrm{d}i_h}{\mathrm{d}t} + 15i_h = 0$$

Notice that this differential equation is the same form as that of RC circuits! If we let $\tau = \frac{L}{R_{TH}} = \frac{1}{15}$, our solution will be identical:

$$i_h(t) = A_1 \mathrm{e}^{-\frac{t}{\tau}} = A_1 \mathrm{e}^{-15t}$$

Now, let's find a particular solution. Let $i_p(t)$ be a particular solution to our differential equation. We can either use the integration factor method or directly use the provided integral (which is derived from using the integrating factor). Here, we will use the integral directly (note that the bounds will not matter in the end since the extra constant term from the integral will combine with the homogeneous solution's arbitrary constant A_1 so you could just evaluate the integral as an indefinite integral, but we will evaluate it as a definite integral from 0 to t for formality):

In our case, $s = \frac{1}{\tau} = 15$ and $b(t) = \frac{1}{2}$, our input term.

$$\begin{aligned} i_p(t) &= \mathrm{e}^{-15t} \int_0^t \mathrm{e}^{15t'} \left(\frac{1}{2}\right) dt' \\ &= \mathrm{e}^{-15t} \left[\frac{1}{30}\mathrm{e}^{15t'}\right]_0^t \\ &= \mathrm{e}^{-15t} \left[\frac{1}{30}\mathrm{e}^{15t} - \frac{1}{30}\right] \\ &= \frac{1}{30} - \frac{1}{30}\mathrm{e}^{-15t} \end{aligned}$$

Now, we can combine the two solutions to get our overall solution.

$$\begin{aligned} i_L(t) &= i_h(t) + i_p(t) \\ &= A_1\mathrm{e}^{-15t} + \frac{1}{30} - \frac{1}{30}\mathrm{e}^{-15t} \\ &= \left(A_1 - \frac{1}{30}\right)\mathrm{e}^{-15t} + \frac{1}{30} \\ &= A\mathrm{e}^{-15t} + \frac{1}{30} \end{aligned}$$

We have defined $A = A_1 - \frac{1}{30}$, which is simply another version of the same arbitrary constant that accounts for our initial condition (this is why the bounds of the integral are not necessary to solve the problem).

Now, we use our initial condition to solve for A.

$$\begin{aligned} i_L(0) = A\mathrm{e}^{-15(0)} + \frac{1}{30} &= 0 \\ A &= -\frac{1}{30} \end{aligned}$$

© UCB EECS 16B, Spring 2023. All Rights Reserved. This may not be publicly shared without explicit permission.

Thus, our final solution is

$$i_L(t) = -\frac{1}{30}e^{-15t} + \frac{1}{30} = \frac{1}{30}\left(1 - e^{-15t}\right)$$

Contributors:

- Anish Muthali.
- Neelesh Ramachandran.
- Kumar Krishna Agrawal.
- Nikhil Jain.

© UCB EECS 16B, Spring 2023. All Rights Reserved. This may not be publicly shared without explicit permission.