Discussion 3A

Note 3 along with corresponding lectures are most relevant to this discussion worksheet.

1. Analyzing a Second-Order Circuit (Adapted from Hambley Example 4.5)

A DC source is connected to a series RLC circuit by a switch that closes at $t = 0$ as shown in Figure 3. The initial conditions are $i(0) = 0$ and $v_C(0) = 0$.

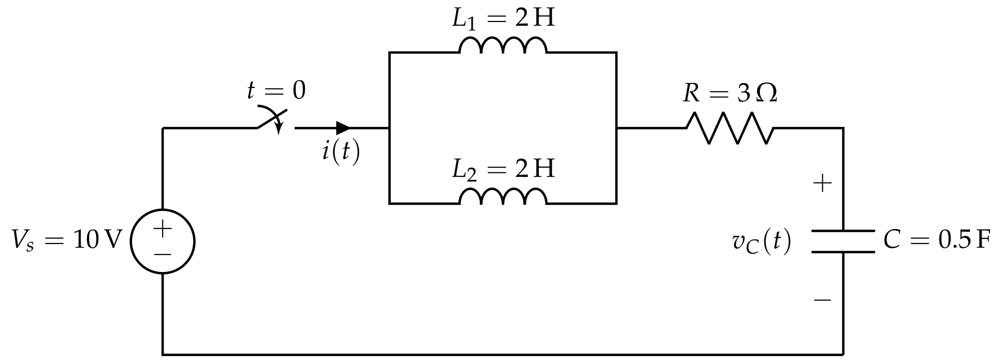


Figure 1: RLC Circuit

(a) **Find the equivalent inductance and redraw the circuit as a standard series RLC.**

Solution: Recall that the equivalent inductance of two inductors in parallel is given by $L_{\text{eq}} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1}$
Therefore,

$$L = \left(\frac{1}{2} + \frac{1}{2}\right)^{-1} = 1\,\text{H} \tag{1}$$

The resulting circuit is as follows:

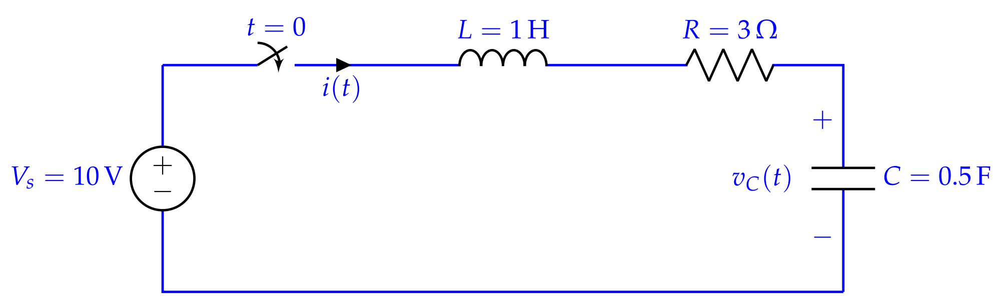


Figure 2: RLC Circuit

(b) **Write the differential equation for $v_c(t)$**

Solution: First, we can write an express for the current in terms of the voltage acorss the capacitance:

$$i(t) = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} \tag{2}$$

Then, writing a KVL equation for the circuit, we have:

$$v_L(t) + v_R(t) + v_C(t) = V_s \tag{3}$$

$$L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + Ri(t) + v_C(t) = V_s \tag{4}$$

$$\tag{5}$$

Substituting in the expression for current $i(t)$, we get:

$$LC\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2} + RC\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + v_C(t) = V_s \tag{6}$$

$$\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + \frac{1}{LC}v_C(t) = \frac{V_s}{LC} \tag{7}$$

(c) **Solve for $v_C(t)$ if $R = 3\,\Omega$.**

Solution:

Step 1: Solve for the homogeneous solution:

Our equation is in the form:

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + 2\frac{\zeta}{\tau}\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \frac{1}{\tau^2}x(t) = f(t) \tag{8}$$

where $f(t) = \frac{V_s}{LC}$ which is a constant. Thus, we know that our solution for $v_C(t)$ will be a combination of the particular solution $v_{Cp}(t)$ and the complementary solution $v_{Cc}(t)$.

Since we have a DC source, we know that the transient or complementary solution will go to 0 over time. Thus, our current and voltage are steady/constant and we can replace inductors with short circuits and capcitors with open circuits. This leads us to determine that $v_{Cp}(t) = V_s = 10\,\mathrm{V}$.

Next, we will find the complementary solution $v_{Cc}(t)$ or the homogeneous solution of our differential equation. When finding the complementary solution, we will follow the following 3 steps:

i. Determine the damping ratio and roots of the characteristic equation

ii. Select the appropriate form for the homogeneous solution, dpending on the value of the damping ratio

iii. Add the homogeneous solution to the particular solution and determine the values of the coefficients (K_1 and K_2) based on initial conditions.

Here, we have $R = 3\,\Omega$, so

$$\tau = \sqrt{LC} = \frac{1}{\sqrt{2}} \tag{9}$$

and the damping ratio $\zeta = \frac{R\tau}{2L} = \frac{3\sqrt{2}}{4}$. Since $\zeta > 1$, we have an overdamped case. **Important:** We consider the exponential candidate solution for our homogeneous solution (i.e. $v_h = \mathrm{e}^{st}$). Using this, our differential equation decomposes into:

$$s^2\mathrm{e}^{st} + 3s\mathrm{e}^{st} + 2\mathrm{e}^{st} = 0 \tag{10}$$

We can view this equation as being in quadratic form (i.e. use the quadratic formula)!

Solving for the roots of our characteristic equation we have:

$$s_1 = -\frac{\zeta}{\tau} + \frac{1}{\tau}\sqrt{\zeta^2 - 1} \tag{11}$$
$$= -1.5 + \sqrt{2}\sqrt{\frac{18}{16} - 1} \tag{12}$$
$$= -1 \tag{13}$$

and

$$s_2 = -\frac{\zeta}{\tau} - \frac{1}{\tau}\sqrt{\zeta^2 - 1} \tag{14}$$
$$= -1.5 - \sqrt{2}\sqrt{\frac{18}{16} - 1} \tag{15}$$
$$= -2 \tag{16}$$

Step 2: Solve for the particular solution using steady-state analysis (note: refer to part d for more details on this):

We know that the homogeneous solution has the form $K_1 e^{s_1 t} + K_2 e^{s_2 t}$ leading us to have the general solution:

$$v_C(t) = v_{Cp}(t) + v_{Cc}(t) = 10 + K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{17}$$

Step 3: Utilize the initial conditions to solve for solution coefficients:

Now, we will find the values of K_1 and K_2 using the given initial conditions. It is given $v_C(0) = 0\,\text{V}$. This gives us that:

$$10 + K_1 + K_2 = 0 \tag{18}$$

Furthermore, since $i(0) = 0\,\text{A}$ we also know that $i(0) = C\frac{dv_C(0)}{dt}$ and thus $\frac{dv_C(0)}{dt} = 0$. Taking the derivative of Equation 17 and plugging in $t = 0$, we get

$$s_1 K_1 e^{s_1(0)} + s_2 K_2 e^{s_2(0)} = 0 \tag{19}$$
$$s_1 K_1 + s_2 K_2 = 0 \tag{20}$$

Now, solving the systems of equations, we get that $K_1 = -20$ and $K_2 = 10$. Substituting these values into Equation 17, we get our final solution:

$$v_C(t) = 10 - 20e^{-t} + 10e^{-2t} \tag{21}$$

(d) **Redraw the circuit in steady state and find the steady state value for $v_C(t)$.**

Solution: Recall that at steady state, inductors act as shorts and capacitors act as open circuits. Using this knowledge, our redrawn circuit is as followss:

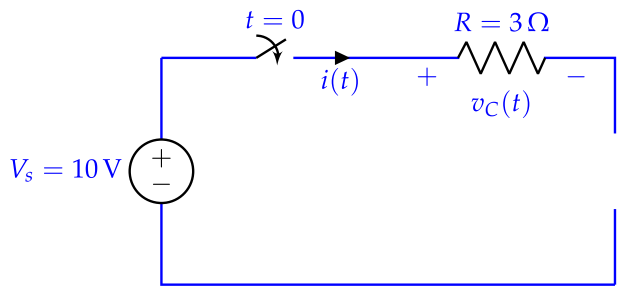
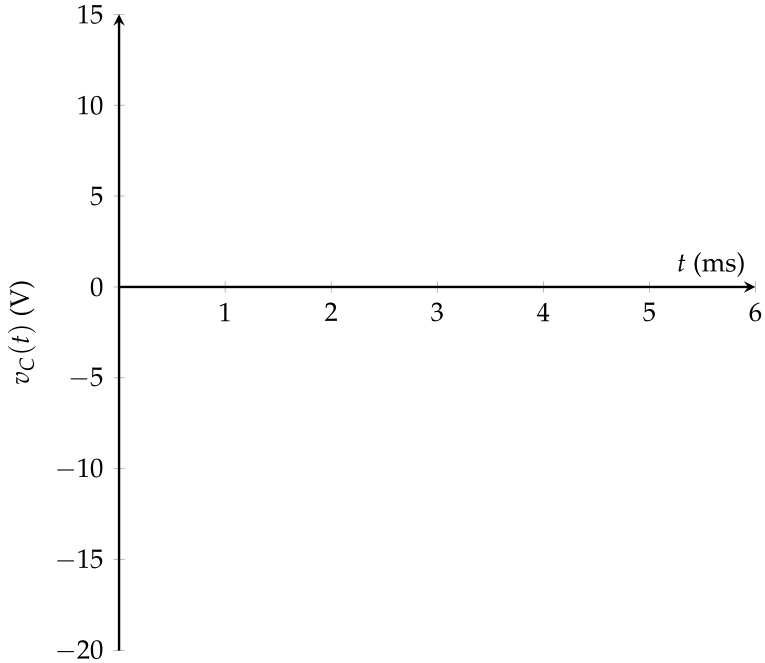


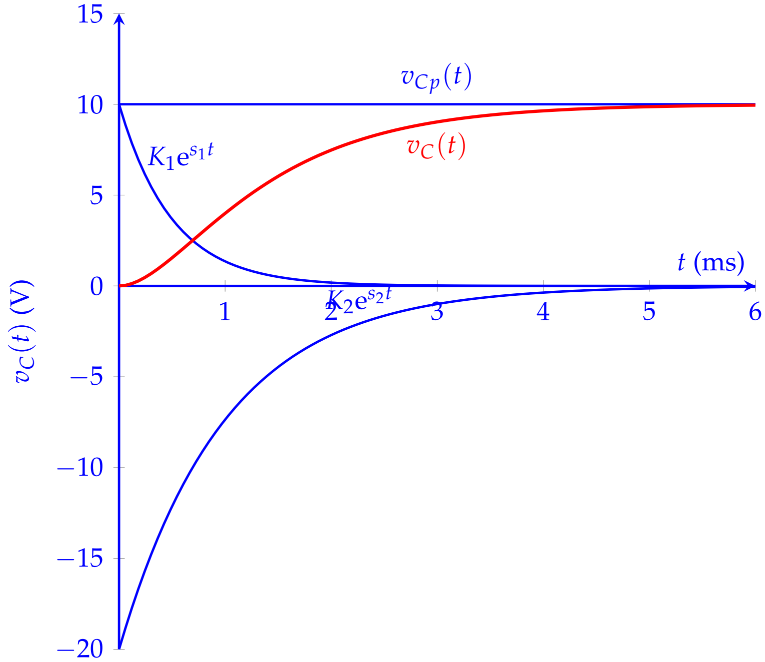
Figure 3: RLC Circuit

Because no current flows into the node with an open circuit, $v_c(t) = V_S = 10\,\text{V}$.

(e) **Plot the equation you calculated for $v_C(t)$.** It may be helpful to draw out each term in your general solution and then add them together.



Solution:



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