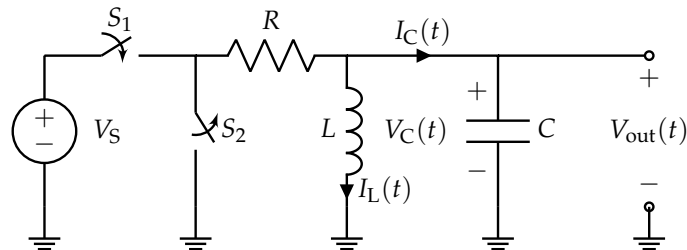


## Discussion 4A

## 1. RLC Circuit from Time to Frequency

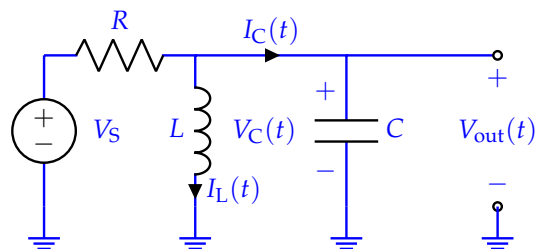
Consider the following circuit fed by a constant voltage source  $V_S$ .



The switch  $S_1$ , open for  $t < 0$ , closes at  $t = 0$ , and the switch  $S_2$ , closed for  $t < 0$ , opens at  $t = 0$ . Assume  $V_C(0) = 0$  and  $I_L(0) = 0$ .

- (a) Derive a set of two differential equations, one for  $I_L(t)$ , the current through the inductor, and one for  $V_C(t)$ , the voltage across the capacitor. Write your answer in terms of  $R$ ,  $L$ ,  $C$ ,  $V_S$ , and constants.

**Solution:** The circuit appears as follows:



From Ohm's law for the resistor and KCL at the node with voltage  $V_C$ , we have:

$$\frac{V_S - V_C}{R} = I_L + I_C \quad (1)$$

Substituting in the I-V relationship for capacitors in the previous equation, we now have:

$$\frac{V_S - V_C}{R} = I_L + C \frac{dV_C}{dt} \quad (2)$$

$$\frac{V_S}{RC} - \frac{V_C}{RC} = \frac{I_L}{C} + \frac{dV_C}{dt} \quad (3)$$

$$\frac{V_S}{RC} - \frac{V_C}{RC} - \frac{I_L}{C} = \frac{dV_C}{dt} \quad (4)$$

Now, we can notice that  $V_C = V_L$  as the inductor and capacitor are in parallel. From the inductor I-V relationship, we have:

$$L \frac{dI_L}{dt} = V_L = V_C \quad (5)$$

$$\frac{dI_L}{dt} = \frac{V_C}{L} \quad (6)$$

In summary, the two differential equations are as follows.

$$\frac{dV_C}{dt} = -\frac{V_C}{RC} - \frac{I_L}{C} + \frac{V_S}{RC} \quad (7)$$

$$\frac{dI_L}{dt} = \frac{V_C}{L} \quad (8)$$

- (b) Using your answers from the previous part, create a vector differential equation with the state vector being  $\vec{x}(t) = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$ . Write your answers in terms of  $R, L, C, V_S$ , and constants.

**Solution:** The previous part has the following differential equations.

$$\frac{dV_C}{dt} = -\frac{V_C}{RC} - \frac{I_C}{C} + \frac{V_S}{RC} \quad (9)$$

$$\frac{dI_L}{dt} = \frac{V_C}{L} \quad (10)$$

Stacking the above equations into matrix-vector form, we have

$$\frac{d}{dt} \vec{x}(t) = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} V_S \quad (11)$$

- (c) Regardless of your answer to the previous part, suppose the vector differential equation is given by

$$\frac{d}{dt} \vec{x}(t) = \underbrace{\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix}}_A \vec{x}(t) + \underbrace{\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{\vec{b}} V_S \quad (12)$$

**First, find the eigenvalues of the matrix A.**

**Solution:** To find our eigenvalues, we use the method used in EECS 16A:

$$A\vec{v} = \lambda\vec{v} \quad (13)$$

$$A\vec{v} - \lambda\vec{v} = 0 \quad (14)$$

$$(A - \lambda I_2)\vec{v} = 0 \quad (15)$$

$$\det(A - \lambda I_2) = 0 \quad (16)$$

$$\det\left(\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0 \quad (17)$$

$$\det\left(\begin{bmatrix} -4 - \lambda & -6 \\ \frac{1}{2} & -\lambda \end{bmatrix}\right) = 0 \quad (18)$$

$$\lambda^2 + 4\lambda + 3 = 0 \quad (19)$$

Therefore our eigenvalues are  $\lambda_1 = -3$  and  $\lambda_2 = -1$ .

- (d) Next, find the eigenvectors that will form your  $V$  basis.

**Solution:** Recall, that the eigenvalues of  $A$  are  $\lambda_1 = -3$  and  $\lambda_2 = -1$ . We find the corresponding eigenvectors by plugging in our eigenvalues:

For  $\lambda_1 = -3$ :

$$\left( \begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \right) \vec{v}_1 = 0 \quad (20)$$

$$\left( \begin{bmatrix} -1 & -6 \\ \frac{1}{2} & 3 \end{bmatrix} \right) \vec{v}_1 = 0 \quad (21)$$

From inspection or Gaussian Elimination, we find that  $\vec{v}_1 = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$ .

For  $\lambda_1 = -1$ :

$$\left( \begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \vec{v}_2 = 0 \quad (22)$$

$$\left( \begin{bmatrix} -3 & -6 \\ \frac{1}{2} & 1 \end{bmatrix} \right) \vec{v}_2 = 0 \quad (23)$$

$$(24)$$

From inspection or Gaussian Elimination, we find that  $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ .

(e) **Now, in order to diagonalize the system, write  $A$  in terms of  $V$ ,  $V^{-1}$ , and  $\Lambda$ .**

**Solution:** From Note 4, we know that:

$$\underbrace{\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} -6 & -2 \\ 1 & 1 \end{bmatrix}}_V \underbrace{\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}}_\Lambda \underbrace{\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix}}_{V^{-1}} \quad (25)$$

(f) **With  $\vec{x}(0) = \vec{0}$ , solve for  $\vec{x}(t)$  and find the asymptotic/steady-state behavior as  $t \rightarrow \infty$ .** **So-**

**lution:** We can define  $\tilde{\vec{x}}(t)$  to be the representation of  $\vec{x}(t)$  in  $V$ -basis. In other words,  $\tilde{\vec{x}}(t) = V^{-1}\vec{x}(t)$  and  $\vec{x}(t) = V\tilde{\vec{x}}(t)$ . Applying this, we have:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}V_S \quad (26)$$

$$V^{-1} \left( \frac{d}{dt}\vec{x}(t) \right) = \underbrace{V^{-1}AV}_{\Lambda} \tilde{\vec{x}}(t) + V^{-1}\vec{b}V_S \quad (27)$$

$$\frac{d}{dt}\tilde{\vec{x}}(t) = \Lambda\tilde{\vec{x}}(t) + \underbrace{\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{V^{-1}\vec{b}} V_S \quad (28)$$

$$\frac{d}{dt}\tilde{\vec{x}}(t) = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \tilde{\vec{x}}(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} V_S \quad (29)$$

The initial condition is  $\tilde{\vec{x}}(0) = V^{-1}\vec{x}(0) = V^{-1}\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . We can solve these equations one row at a time, either by using substitution or the general first order differential equation solution. We

use the latter approach. Solve the first row equation for  $\tilde{x}_1(t)$ .

$$\frac{d}{dt}\tilde{x}_1 = -3\tilde{x}_1 - V_S \quad (30)$$

$$\implies \tilde{x}_1(t) = -V_S e^{-3t} \int_0^t e^{3\theta} d\theta \quad (31)$$

$$= -V_S e^{-3t} \frac{e^{3\theta}}{3} \Big|_{\theta=0}^{\theta=t} \quad (32)$$

$$= -V_S e^{-3t} \frac{e^{3t} - 1}{3} \quad (33)$$

$$= V_S \frac{e^{-3t} - 1}{3} \quad (34)$$

And now solve the second row equation for  $\tilde{x}_2(t)$ .

$$\frac{d}{dt}\tilde{x}_2 = -\tilde{x}_2 + V_S \quad (35)$$

$$\implies \tilde{x}_2(t) = V_S e^{-t} \int_0^t e^{\theta} d\theta \quad (36)$$

$$= V_S e^{-t} (e^t - 1) \quad (37)$$

$$= V_S (1 - e^{-t}) \quad (38)$$

so

$$\vec{\tilde{x}}(t) = V_S \begin{bmatrix} \frac{e^{-3t}-1}{3} \\ 1 - e^{-t} \end{bmatrix} \quad (39)$$

Therefore,

$$\vec{x}(t) = V \vec{\tilde{x}}(t) = V_S \begin{bmatrix} -6 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{e^{-3t}-1}{3} \\ 1 - e^{-t} \end{bmatrix} = V_S \begin{bmatrix} 2(e^{-t} - e^{-3t}) \\ \frac{e^{-3t} - 3e^{-t} + 2}{3} \end{bmatrix} \quad (40)$$

Taking a limit as  $t \rightarrow \infty$ , we have

$$\lim_{t \rightarrow \infty} \vec{x}(t) = V_S \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} \quad (41)$$

In particular, we can say that  $\lim_{t \rightarrow \infty} V_C(t) = 0$ , and that  $\lim_{t \rightarrow \infty} I_L(t) = \frac{2}{3} V_S$ .