

## 1. Transfer Function Practice

Transfer functions take an input phasor and “transform” it into an output phasor. Most of the work we will do with transfer functions is analyzing how it will “respond” to a specific kind of input. We will also design our own transfer functions using common circuit components such as resistors, inductors, and capacitors to achieve some specified behavior. A block diagram of a transfer function is represented below. In this discussion, we will learn how to derive  $H(j\omega)$  from a given circuit, and we will analyze how it behaves for certain values of  $\omega$ .

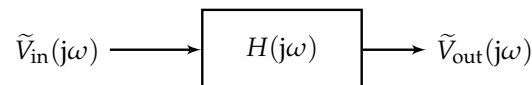


Figure 1: Transfer Function Block Diagram

Recall that  $Z_L = j\omega L$  and  $Z_C = \frac{1}{j\omega C}$ . For large  $\omega$ ,  $|Z_L| = \omega L$  becomes large (and becomes small for small  $\omega$ ). On the other hand, for large  $\omega$ ,  $|Z_C| = \frac{1}{\omega C}$  becomes small (and becomes large for small  $\omega$ ).

In this problem, you’ll be deriving some transfer functions. For each circuit:

- Determine the **transfer function**  $H(j\omega) = \frac{\tilde{V}_{\text{out}}(j\omega)}{\tilde{V}_{\text{in}}(j\omega)}$ .
- How does  $|H(j\omega)|$  respond as  $\omega \rightarrow 0$  (**low frequencies**) and as  $\omega \rightarrow \infty$  (**high frequencies**)?
- Is the circuit a **high-pass filter, low-pass filter, or band-pass filter**?
- For parts (a) and (b), find the **cutoff frequency**  $\omega_c$ , which is the frequency such that

$$|H(j\omega_c)| = \frac{|H(j\omega_c)|_{\text{max}}}{\sqrt{2}} \quad (1)$$

- Use the provided element values to **sketch the magnitude** ( $|H(j\omega)|_{dB} = 20 \log_{10}(|H(j\omega)|)$ ) **and phase** ( $\angle H(j\omega)$ ) **Bode plots** (on the provided plots) of  $H(j\omega)$  (to help you do this, you can either use your knowledge of Bode plots for basic filters along with the cutoff frequency  $\omega_c$ , or think about the zero and pole frequencies for each transfer function).

### Poles and Zeros:

Note: This is brief introduction to a concept (covered more in depth in future classes) that will help us think about Bode plots, especially for more complicated transfer functions. For simple filters/transfer functions, you should be able to recognize the Bode plot whether you think about poles/zeros or not.

Suppose we write the transfer function in the following form:

$$H(j\omega) = A \frac{\prod_{n=1}^N (\omega_{zn} + j\omega)}{\prod_{m=1}^M (\omega_{pm} + j\omega)} = A \frac{(\omega_{z1} + j\omega) \cdots (\omega_{zN} + j\omega)}{(\omega_{p1} + j\omega) \cdots (\omega_{pM} + j\omega)} \quad (2)$$

where each  $\omega_z$  is called a zero frequency and each  $\omega_p$  is called a pole frequency.

We can find the magnitude and phase of the transfer function using complex algebra as such:

$$|H(j\omega)| = A \frac{\prod_{n=1}^N |\omega_{zn} + j\omega|}{\prod_{m=1}^M |\omega_{pm} + j\omega|} \quad (3)$$

$$\angle H(j\omega) = \sum \angle(\omega_z + j\omega) - \sum \angle(\omega_p + j\omega) \quad (4)$$

Now, let's look at pole/zero factor separately.

Let  $z = \omega_{zp} + j\omega$ . We can calculate  $|z| = \sqrt{\omega_{zp}^2 + \omega^2}$  and  $\angle z = \text{atan2}(\omega, \omega_{zp})$ .

When we create Bode plots, we like to approximate these values. For  $|z|$ :

- When  $\omega < \omega_{zp}$ ,  $|z| \approx \sqrt{\omega_{zp}^2} = \omega_{zp}$ .
- When  $\omega > \omega_{zp}$ ,  $|z| \approx \sqrt{\omega^2} = \omega$ .

For  $\angle z$ :

- When  $\omega < \frac{1}{10}\omega_{zp}$ ,  $\angle z \approx 0$ .
- When  $\omega > 10\omega_{zp}$ ,  $\angle z \approx \frac{\pi}{2}$ .
- When  $\frac{1}{10}\omega_{zp} < \omega < 10\omega_{zp}$ ,  $\angle z$  transitions by an approximately constant phase per decade.

These mathematical approximations create some general rules for Bode plots. For magnitude plots:

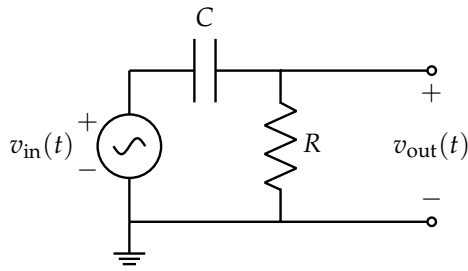
- When our Bode plot passes a **zero** frequency, the slope **increases by**  $20 \frac{\text{dB}}{\text{dec}}$ .
- When our Bode plot passes a **pole** frequency, the slope **decreases by**  $20 \frac{\text{dB}}{\text{dec}}$ .

For phase plots:

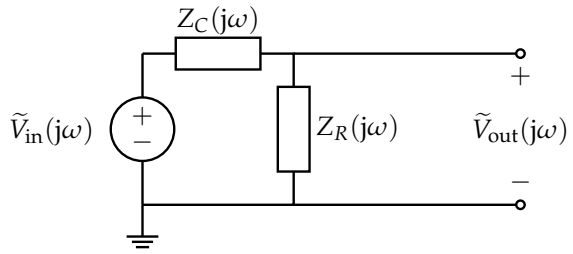
- When our Bode plot passes a **zero** frequency, the phase **increases by**  $\frac{\pi}{2} / 90$  **degrees** from one decade below to one decade above the zero frequency.
- When our Bode plot passes a **pole** frequency, the phase **decreases by**  $\frac{\pi}{2} / 90$  **degrees** from one decade below to one decade above the pole frequency.

Again, this is not a comprehensive overview of poles and zeros (for example, you could have  $\omega_{zp} - j\omega$  factors); it is meant as an introduction to the concept to provide some mathematical background on why we plot Bode plots the way we do.

(a) **RC circuit** ( $R = 1\text{ k}\Omega$ ,  $C = 1\text{ }\mu\text{F}$ ):

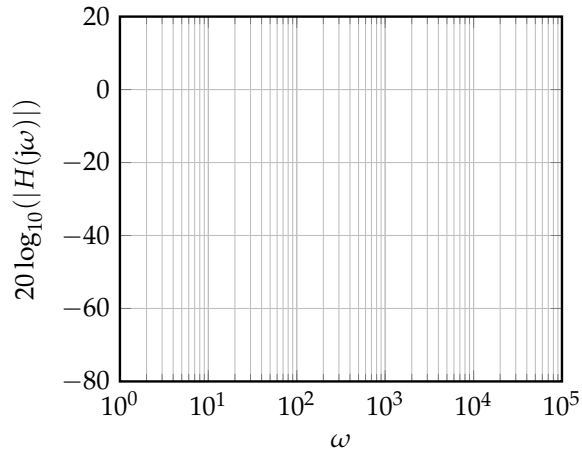


(a) Circuit in "time domain"

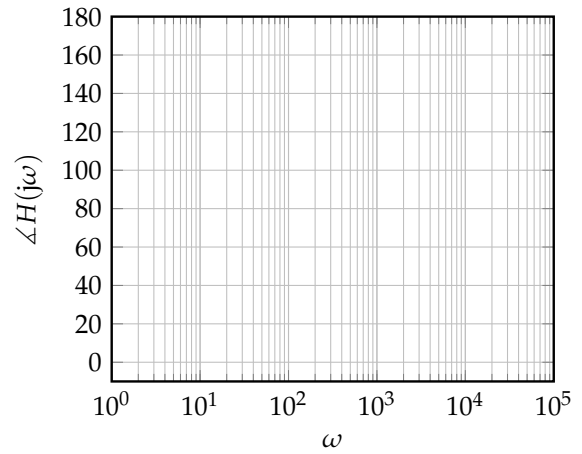


(b) Circuit in "phasor domain"

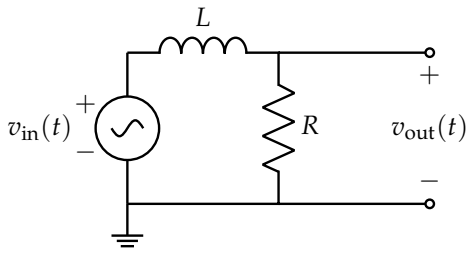
Plot of  $|H(j\omega)|_{dB}$  (for you to draw).



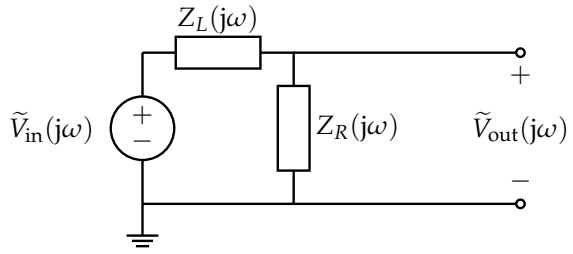
Plot of  $\angle H(j\omega)$  (for you to draw).



(b) **LR circuit** ( $L = 5 \text{ H}$ ,  $R = 500 \Omega$ ):

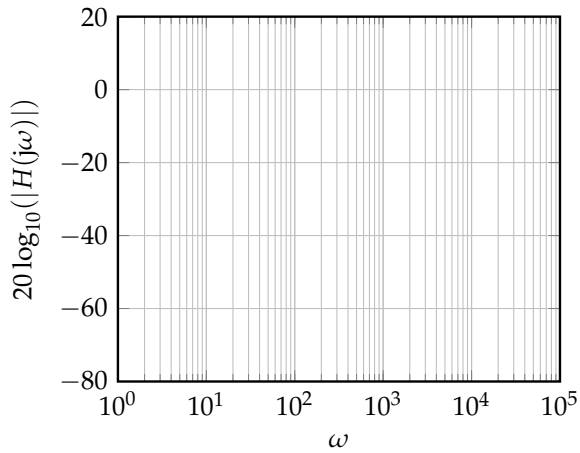


(a) Circuit in "time domain"

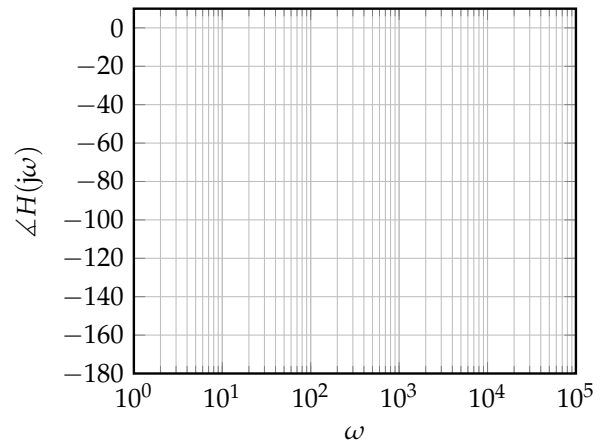


(b) Circuit in "phasor domain"

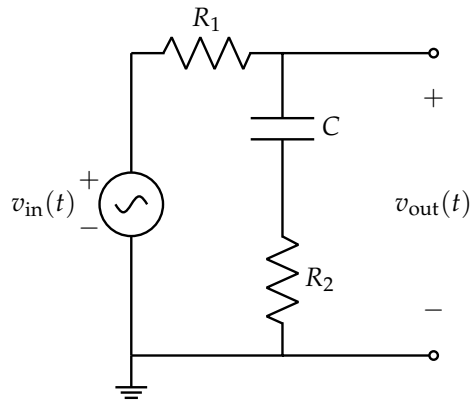
Plot of  $|H(j\omega)|_{dB}$  (for you to draw).



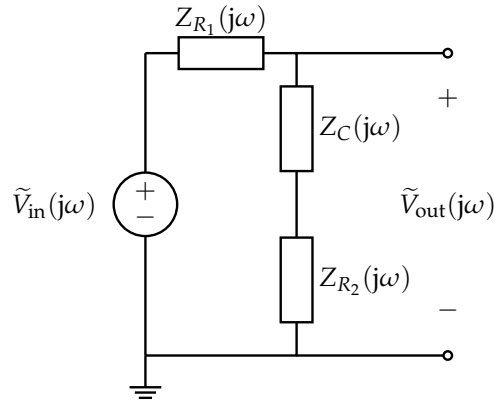
Plot of  $\angle H(j\omega)$  (for you to draw).



(c) (PRACTICE) RCR circuit ( $R_1 = 9 \text{ k}\Omega$ ,  $R_2 = 1 \text{ k}\Omega$ ,  $C = 1 \text{ }\mu\text{F}$ ):

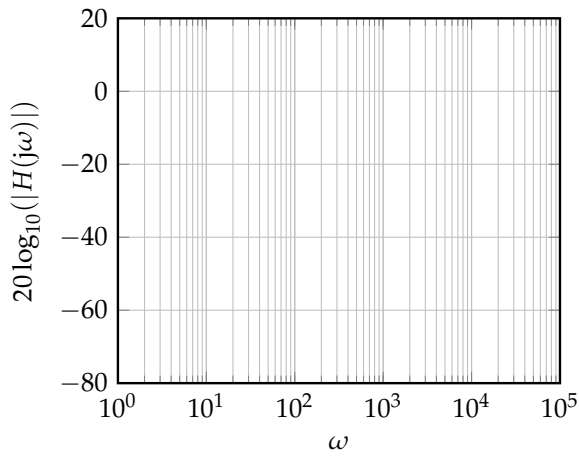


(a) Circuit in "time domain"

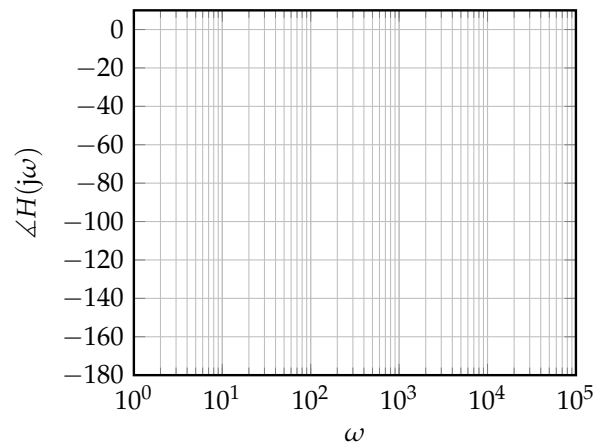


(b) Circuit in "phasor domain"

Plot of  $|H(j\omega)|_{dB}$  (for you to draw).



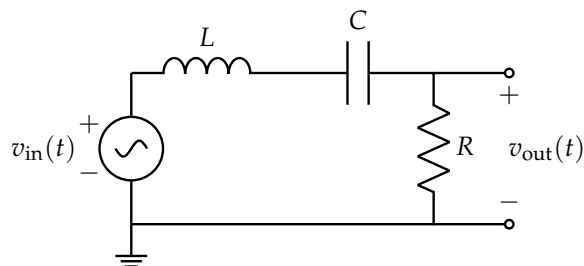
Plot of  $\angle H(j\omega)$  (for you to draw).



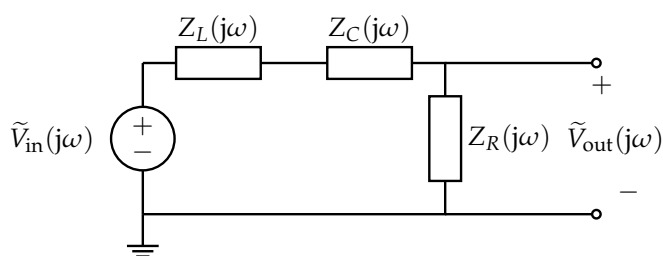
- (d) **Assuming**  $v_{\text{in}}(t) = 12 \sin(\omega_{\text{in}}t)$  **compute the**  $v_{\text{out}}(t)$  **using the transfer function computed in part 1.a.** Remember that  $R = 1 \text{ k}\Omega$  and  $C = 1 \text{ }\mu\text{F}$  for this circuit, and assume  $\omega_{\text{in}} = 1000 \frac{\text{rad}}{\text{s}}$ . In words, what is the effect of the transfer function in part 1.a on the magnitude and phase of the input signal?

## 2. (OPTIONAL) Band-Pass Filter

It is quite common to need to design a filter which selects only a narrow range of frequencies. One example is in WiFi radios, it is desirable to select only the 2.4GHz frequency containing your data, and reject information from other nearby cellular or bluetooth frequencies. This type of filter is called a band-pass filter; we will explore the design of this type of filter in this problem.



(a) Circuit in "time domain"



(b) Circuit in "phasor domain"

(a) Write down the transfer function  $H(j\omega) = \frac{\tilde{V}_{out}(j\omega)}{\tilde{V}_{in}(j\omega)}$  for this circuit.

(b) Consider the inductor, capacitor, and resistor connected in series. Write down the impedance of the series RLC combination in the form  $Z_{RLC}(j\omega) = A(\omega) + jX(\omega)$ , where  $A(\omega)$  and  $X(\omega)$  are real valued functions of  $\omega$ . At what frequency  $\omega_r$  does  $X(\omega_r) = 0$ ? (i.e. at what frequency is the impedance of the series combination of RLC purely real — meaning that the imaginary terms coming from the capacitor and inductor completely cancel each other. This is called the *resonant frequency*.)

- (c) **Find an expression for  $|H(j\omega)|$ . What is  $|H(j\omega_r)|$ ? What is  $|H(j\omega_r/10)|$ ? What is  $|H(j10\omega_r)|$ ?**  
Rank the three quantities:  $|H(j\omega_r)|$ ,  $|H(j\omega_r/10)|$ ,  $|H(j10\omega_r)|$ . What do you think the magnitude plot looks like?

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