EECS 16B Designing Information Systems and Devices II UC Berkeley Spring 2023

Discussion 5B

1. Transfer Function Practice

Transfer functions take an input phasor and "transform" it into an output phasor. Most of the work we will do with transfer functions is analyzing how it will "respond" to a specific kind of input. We will also design our own transfer functions using common circuit components such as resistors, inductors, and capacitors to achieve some specified behavior. A block diagram of a transfer function is represented below. In this discussion, we will learn how to derive $H(\mathrm{j}\omega)$ from a given circuit, and we will analyze how it behaves for certain values of ω.

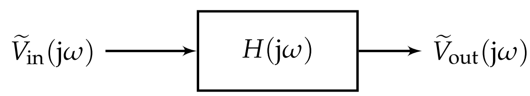


Figure 1: Transfer Function Block Diagram

Recall that $Z_L = \mathrm{j}\omega L$ and $Z_C = \frac{1}{\mathrm{j}\omega C}$. For large ω, $|Z_L| = \omega L$ becomes large (and becomes small for small ω). On the other hand, for large ω, $|Z_C| = \frac{1}{\omega C}$ becomes small (and becomes large for small ω).

In this problem, you'll be deriving some transfer functions. For each circuit:

- Determine the **transfer function** $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\mathrm{j}\omega)}{\widetilde{V}_{\text{in}}(\mathrm{j}\omega)}$.
- How does $|H(\mathrm{j}\omega)|$ respond as $\omega \to 0$ **(low frequencies)** and as $\omega \to \infty$ **(high frequencies)**?
- Is the circuit a **high-pass filter, low-pass filter, or band-pass filter**?
- **For parts (a) and (b)**, find the **cutoff frequency** ω_c, which is the frequency such that

$$|H(\mathrm{j}\omega_c)| = \frac{|H(\mathrm{j}\omega_c)|_{\max}}{\sqrt{2}} \tag{1}$$

- Use the provided element values to **sketch the magnitude (**$|H(\mathrm{j}\omega)|_{dB} = 20\log_{10}(|H(\mathrm{j}\omega)|)$**) and phase (**$\measuredangle H(\mathrm{j}\omega)$**) Bode plots** (on the provided plots) of $H(\mathrm{j}\omega)$ (to help you do this, you can either use your knowledge of Bode plots for basic filters along with the cutoff frequency ω_c, or think about the zero and pole frequencies for each transfer function).

Poles and Zeros:

Note: This is brief introduction to a concept (covered more in depth in future classes) that will help us think about Bode plots, especially for more complicated transfer functions. For simple filters/transfer functions, you should be able to recognize the Bode plot whether you think about poles/zeros or not.

Suppose we write the transfer function in the following form:

$$H(\mathrm{j}\omega) = A\frac{\prod_{n=1}^{N}(\omega_{zn} + \mathrm{j}\omega)}{\prod_{m=1}^{M}(\omega_{pm} + \mathrm{j}\omega)} = A\frac{(\omega_{z1} + \mathrm{j}\omega)\cdots(\omega_{zN} + \mathrm{j}\omega)}{(\omega_{p1} + \mathrm{j}\omega)\cdots(\omega_{pM} + \mathrm{j}\omega)} \tag{2}$$

where each ω_z is called a zero frequency and each ω_p is called a pole frequency.

We can find the magnitude and phase of the transfer function using complex algebra as such:

$$|H(\mathrm{j}\omega)| = A\frac{\prod_{n=1}^{N}|\omega_{zn} + \mathrm{j}\omega|}{\prod_{m=1}^{M}|\omega_{pm} + \mathrm{j}\omega|} \tag{3}$$

$$\measuredangle H(\mathrm{j}\omega) = \sum \measuredangle(\omega_z + \mathrm{j}\omega) - \sum \measuredangle(\omega_p + \mathrm{j}\omega) \tag{4}$$

Now, let's look at pole/zero factor separately.

Let $z = \omega_{zp} + \mathrm{j}\omega$. We can calculate $|z| = \sqrt{\omega_{zp}^2 + \omega^2}$ and $\measuredangle z = \mathrm{atan2}(\omega, \omega_{zp})$.

When we create Bode plots, we like to approximate these values. For $|z|$:

- When $\omega < \omega_{zp}$, $|z| \approx \sqrt{\omega_{zp}^2} = \omega_{zp}$.
- When $\omega > \omega_{zp}$, $|z| \approx \sqrt{\omega^2} = \omega$.

For $\measuredangle z$:

- When $\omega < \frac{1}{10}\omega_{zp}$, $\measuredangle z \approx 0$.
- When $\omega > 10\omega_{zp}$, $\measuredangle z \approx \frac{\pi}{2}$.
- When $\frac{1}{10}\omega_{zp} < \omega < 10\omega_{zp}$, $\measuredangle z$ transitions by an approximately constant phase per decade.

These mathematical approximations create some general rules for Bode plots. For magnitude plots:

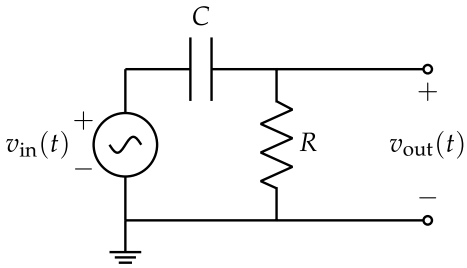
- When our Bode plot passes a **zero** frequency, the slope **increases by** $20\frac{\mathrm{dB}}{\mathrm{dec}}$.
- When our Bode plot passes a **pole** frequency, the slope **decreases by** $20\frac{\mathrm{dB}}{\mathrm{dec}}$.

For phase plots:

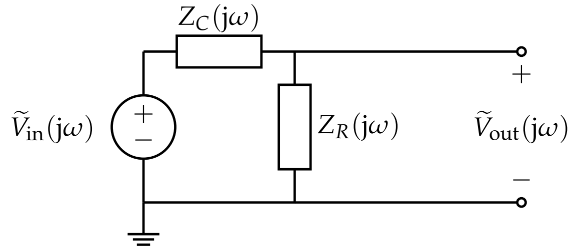
- When our Bode plot passes a **zero** frequency, the phase **increases by** $\frac{\pi}{2}$ **/** 90 **degrees** from one decade below to one decade above the zero frequency.
- When our Bode plot passes a **pole** frequency, the phase **decreases by** $\frac{\pi}{2}$ **/** 90 **degrees** from one decade below to one decade above the pole frequency.

Again, this is not a comprehensive overview of poles and zeros (for example, you could have $\omega_{zp} - \mathrm{j}\omega$ factors); it is meant as an introduction to the concept to provide some mathematical background on why we plot Bode plots the way we do.

(a) **RC circuit** ($R = 1\,\mathrm{k\Omega}$, $C = 1\,\mu\mathrm{F}$):

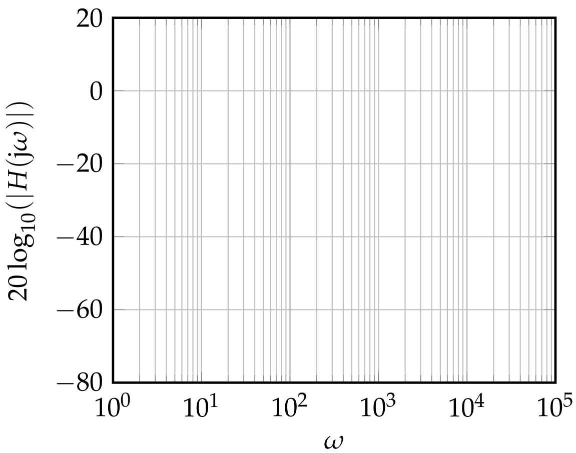


(a) Circuit in "time domain"

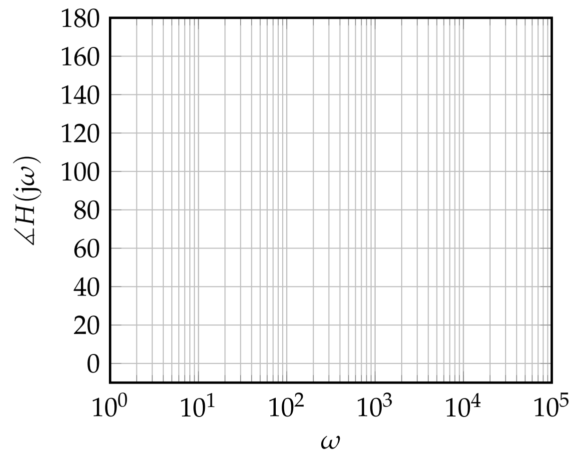


(b) Circuit in "phasor domain"

Plot of $|H(j\omega)|_{dB}$ (for you to draw).



Plot of $\measuredangle H(j\omega)$ (for you to draw).



Solution: We'll use the voltage divider formula to find $\widetilde{V}_{out}(j\omega)$:

$$\widetilde{V}_{out}(j\omega) = \frac{Z_R}{Z_R + Z_C}\widetilde{V}_{in}(j\omega) \tag{5}$$

Recalling the expression for the impendances, we note that for the resistor $Z_R = R$, and for the capacitor $Z_C = \frac{1}{j\omega C}$. Plugging in the impedances gives

$$H(j\omega) = \frac{\widetilde{V}_{out}(j\omega)}{\widetilde{V}_{in}(j\omega)} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC} \tag{6}$$

At low frequencies, we have

$$\lim_{\omega\to 0} |H(j\omega)| = \lim_{\omega\to 0} \frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}} = 0 \tag{7}$$

At high frequencies, we have

$$\begin{aligned} \lim_{\omega\to\infty} |H(j\omega)| &= \lim_{\omega\to\infty} \frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}} &(8)\\ &= \lim_{\omega\to\infty} \frac{\omega RC}{\sqrt{\omega^2 R^2 C^2}} &(9)\\ &= 1 &(10) \end{aligned}$$

So this circuit is a high-pass filter.

For this transfer function, $|H(\mathrm{j}\omega)|_{max} = 1$. Thus, to find the cutoff frequency ω_c, we need to find when $|H(\mathrm{j}\omega_c)| = \frac{1}{\sqrt{2}}$.

$$|H(\mathrm{j}\omega_c)| = \frac{1}{\sqrt{2}} \tag{11}$$

$$\frac{\omega RC}{\sqrt{1+\omega_c^2 R^2 C^2}} = \frac{1}{\sqrt{2}} \tag{12}$$

$$1+\omega_c^2 R^2 C^2 = 2\omega^2 R^2 C^2 \tag{13}$$

$$\omega_c = \frac{1}{RC} \tag{14}$$

$$= \frac{1}{(10^3)(10^{-6})} = 10^3 \frac{\text{rad}}{\text{s}} \tag{15}$$

Notice that this can be observed from the transfer function itself by writing it in the following form:

$$\frac{\mathrm{j}\omega RC}{1+\mathrm{j}\omega RC} = \frac{\mathrm{j}\frac{\omega}{\frac{1}{RC}}}{1+\mathrm{j}\frac{\omega}{\frac{1}{RC}}} = \frac{\mathrm{j}\frac{\omega}{\omega_c}}{1+\mathrm{j}\frac{\omega}{\omega_c}} \tag{16}$$

Now, we can create our Bode plots. We can write our transfer function in the following form:

$$H(\mathrm{j}\omega) = \frac{\mathrm{j}\omega}{\omega_c + \mathrm{j}\omega} \tag{17}$$

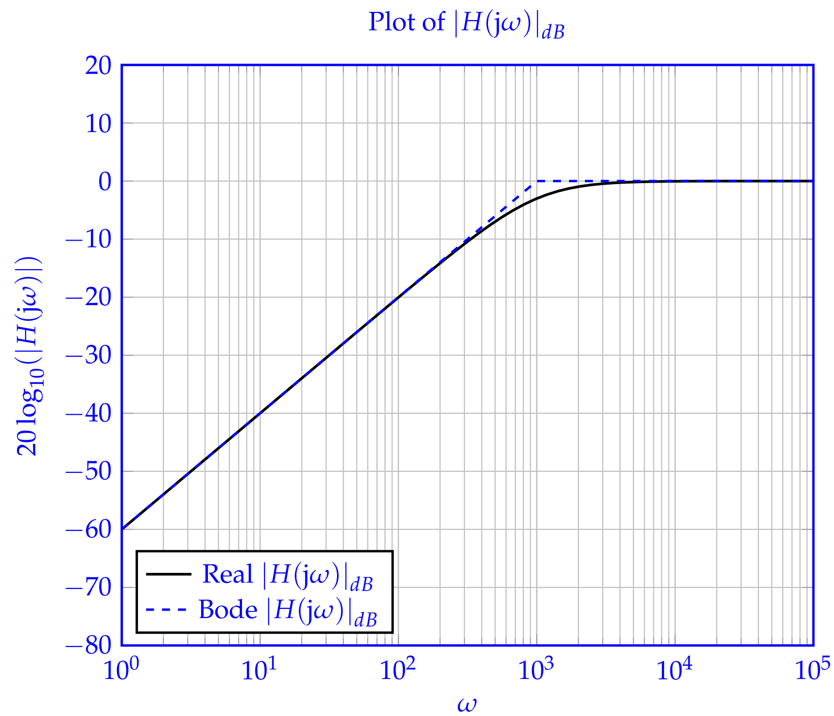
For a simple high pass filter such as this, the transition point is the cutoff frequency ω_c and the shape will be that of a basic high pass filter.

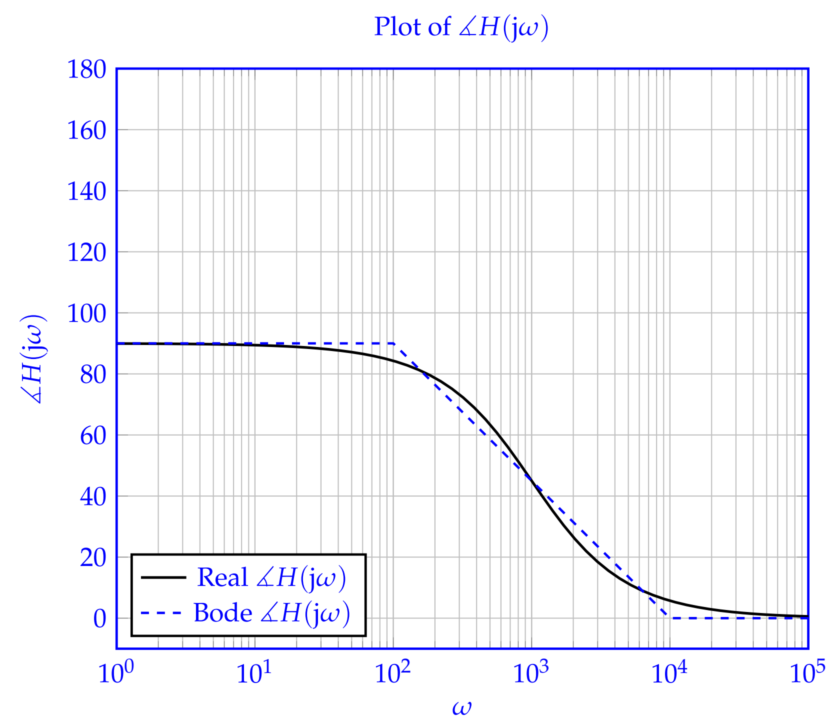
If we use poles and zeros, this transfer function has a zero at $\omega_z = 0$ and a pole at $\omega_p = \omega_c$.

The magnitude plot will start off with a slope of $+20\frac{\text{dB}}{\text{dec}}$ due to $\omega_z = 0$, and at $\omega_p = \omega_c = 10^3$, the slope will decrease to $0\frac{\text{dB}}{\text{dec}}$, at which point the magnitude should be $20\log_{10}(|H(\mathrm{j}\infty)|) = 20\log_{10}(1) = 0\,\text{dB}$.

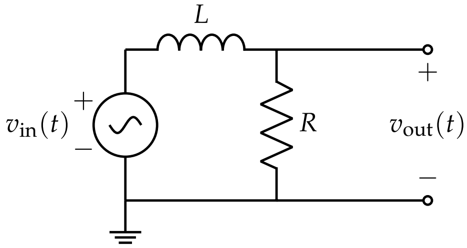
The phase plot will start off at $\frac{\pi}{2}$ due to $\omega_z = 0$ and will decrease linearly in the Bode plot from $\frac{\pi}{2}$ to 0 from 10^2 to 10^4 due to $\omega_p = \omega_c = 10^3$.

The magnitude and phase plots of $H(\mathrm{j}\omega)$ are shown below.

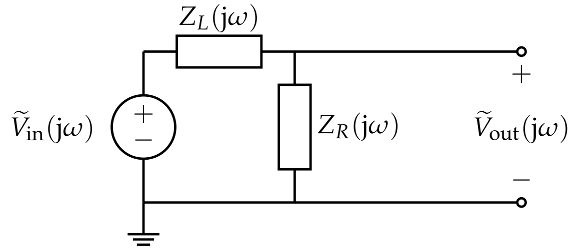

Plot of $|H(j\omega)|_{dB}$
$20\log_{10}(|H(j\omega)|)$
20
10
0
−10
−20
−30
−40
−50
−60
−70
−80
Real $|H(j\omega)|_{dB}$
Bode $|H(j\omega)|_{dB}$
10^0
10^1
10^2
10^3
10^4
10^5
ω


Plot of $\measuredangle H(j\omega)$
$\measuredangle H(j\omega)$
180
160
140
120
100
80
60
40
20
0
Real $\measuredangle H(j\omega)$
Bode $\measuredangle H(j\omega)$
10^0
10^1
10^2
10^3
10^4
10^5
ω

(b) **LR circuit** ($L = 5\,\mathrm{H}$, $R = 500\,\Omega$):

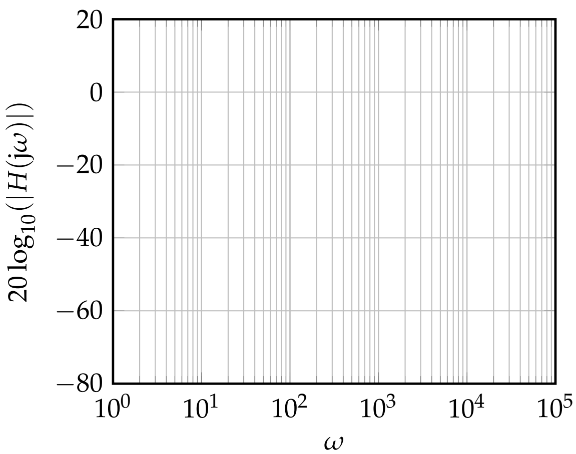


(a) Circuit in "time domain"

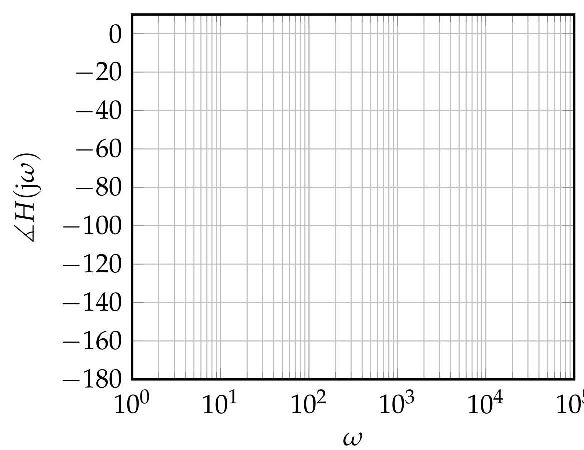


(b) Circuit in "phasor domain"

Plot of $|H(\mathrm{j}\omega)|_{dB}$ (for you to draw).



Plot of $\measuredangle H(\mathrm{j}\omega)$ (for you to draw).



Solution: The strategy is the same as the previous part, using the voltage divider formula, i.e. ,

$$\widetilde{V}_{\text{out}}(\mathrm{j}\omega) = \frac{Z_R}{Z_R + Z_L}\widetilde{V}_{\text{in}}(\mathrm{j}\omega)$$

A similar manipulation to the previous part gives

$$H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\mathrm{j}\omega)}{\widetilde{V}_{\text{in}}(\mathrm{j}\omega)} = \frac{R}{R + \mathrm{j}\omega L} \tag{18}$$

At low frequencies, we have

$$\lim_{\omega \to 0} |H(\mathrm{j}\omega)| = \lim_{\omega \to 0} \frac{R}{\sqrt{R^2 + \omega^2 L^2}} = 1 \tag{19}$$

while at high frequencies, we have

$$\lim_{\omega \to \infty} |H(\mathrm{j}\omega)| = \lim_{\omega \to \infty} \frac{R}{R^2 + \omega^2 L^2} = 0 \tag{20}$$

So this circuit is a low-pass filter. Notice that this circuit resembles the one in the previous part, except we have replaced the capacitor with an inductor.

For this transfer function, $|H(\mathrm{j}\omega)|_{max} = 1$. Thus, to find the cutoff frequency ω_c, we need to find when $|H(\mathrm{j}\omega_c)| = \frac{1}{\sqrt{2}}$.

$$|H(\mathrm{j}\omega_c)| = \frac{1}{\sqrt{2}} \tag{21}$$

$$\frac{R}{\sqrt{R^2 + \omega_c^2 L^2}} = \frac{1}{\sqrt{2}} \tag{22}$$

$$R^2 + \omega_c^2 L^2 = 2R^2 \tag{23}$$

$$\omega_c = \frac{R}{L} \tag{24}$$

$$= \frac{500}{5} = 10^2 \frac{\text{rad}}{\text{s}} \tag{25}$$

Notice that this can be observed from the transfer function itself by writing it in the following form:

$$\frac{R}{R + \mathrm{j}\omega L} = \frac{1}{1 + \mathrm{j}\frac{\omega}{\frac{R}{L}}} = \frac{1}{1 + \mathrm{j}\frac{\omega}{\omega_c}} \tag{26}$$

Now, we can create our Bode plots. We can write our transfer function in the following form:

$$H(\mathrm{j}\omega) = \omega_c \frac{1}{\omega_c + \mathrm{j}\omega} \tag{27}$$

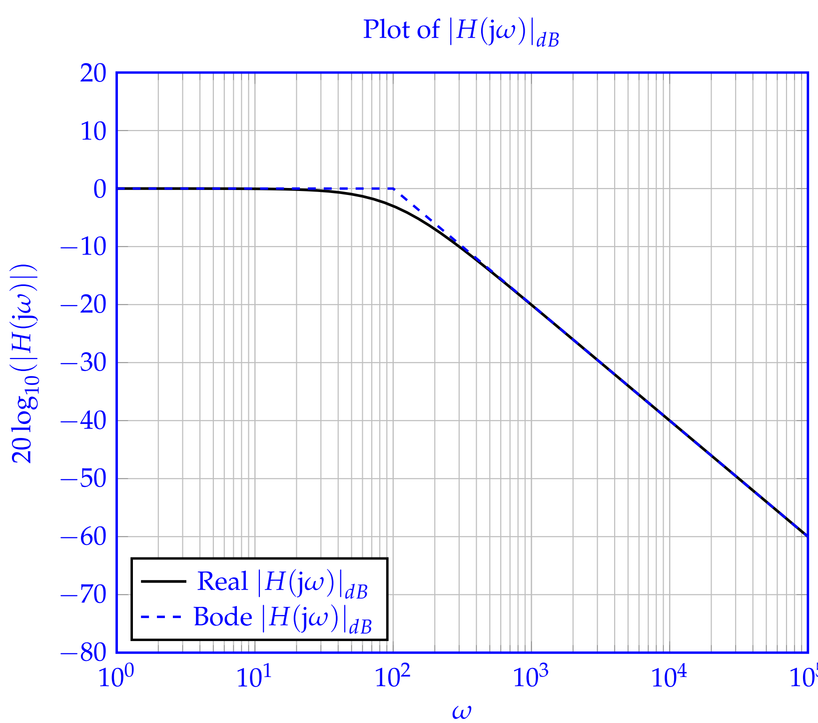
For a simple low pass filter such as this, the transition point is the cutoff frequency ω_c and the shape will be that of a basic low pass filter.

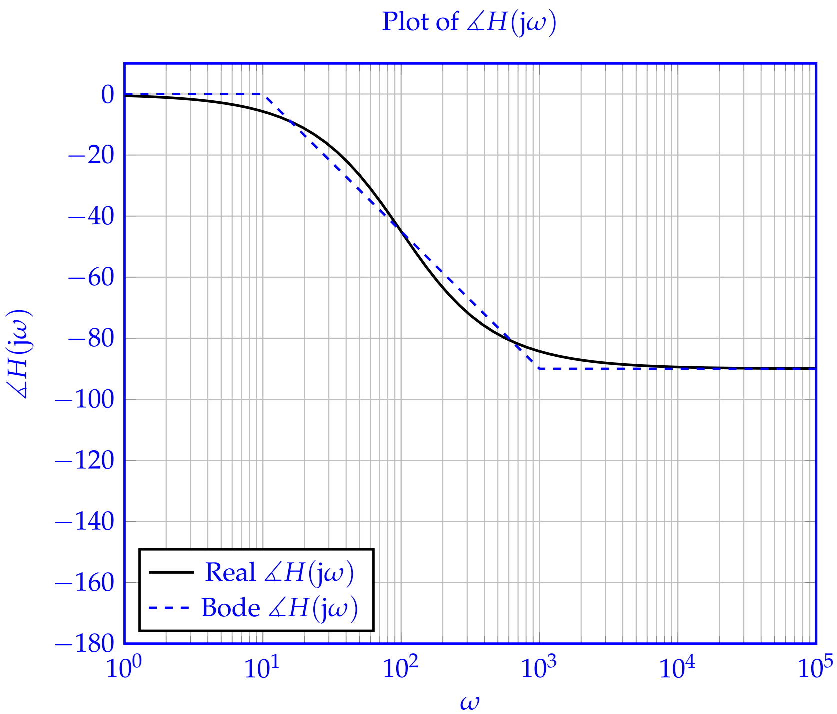
If we use poles and zeros, this transfer function has a pole at $\omega_p = \omega_c$.

The magnitude plot will start off at $20\log_{10}(|H(\mathrm{j}0)|) = 20\log_{10}(1) = 0\,\mathrm{dB}$ and at $\omega_p = \omega_c = 10^2$, the slope will decrease to $-20\frac{\mathrm{dB}}{\mathrm{dec}}$.

The phase plot will start off at 0 and will decrease linearly in the Bode plot from 0 to $-\frac{\pi}{2}$ from 10^1 to 10^3 due to $\omega_p = \omega_c = 10^2$.

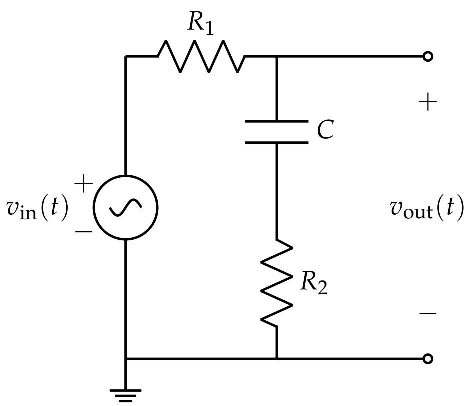
The magnitude and phase plots of $H(\mathrm{j}\omega)$ are shown below.



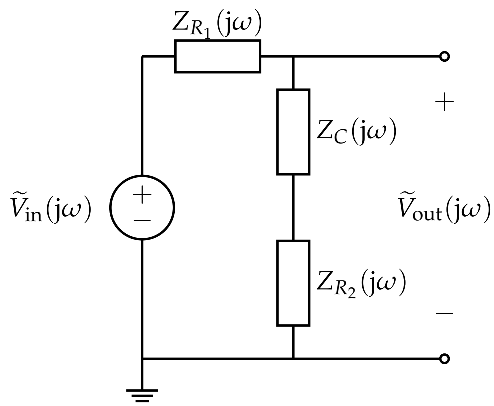


Plot of $\measuredangle H(j\omega)$
0
−20
−40
−60
−80
−100
−120
−140
−160
−180
$\measuredangle H(j\omega)$
Real $\measuredangle H(j\omega)$
Bode $\measuredangle H(j\omega)$
10^0
10^1
10^2
10^3
10^4
10^5
ω

(c) **(PRACTICE) RCR circuit** ($R_1 = 9\,\text{k}\Omega$, $R_2 = 1\,\text{k}\Omega$, $C = 1\,\mu\text{F}$):

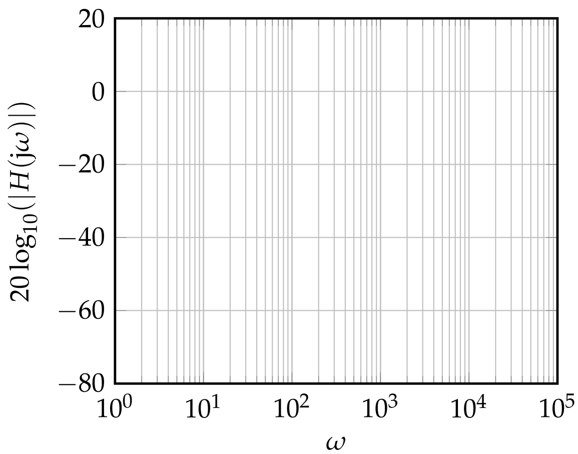


(a) Circuit in "time domain"

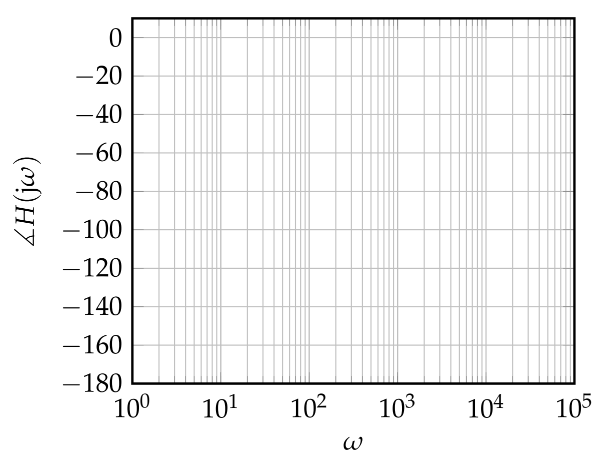


(b) Circuit in "phasor domain"

Plot of $|H(\mathrm{j}\omega)|_{dB}$ (for you to draw).



Plot of $\measuredangle H(\mathrm{j}\omega)$ (for you to draw).



Solution: Even though there are three components instead of two, we can still use the voltage divider formula by treating R_2 and C as a single impedance given by $Z = Z_C + Z_{R_2}$, giving us $Z = R_2 + \frac{1}{\mathrm{j}\omega C}$. This would give us

$$\widetilde{V}_{\text{out}}(\mathrm{j}\omega) = \frac{Z}{Z_{R_1} + Z}\widetilde{V}_{\text{in}}(\mathrm{j}\omega) \tag{28}$$

Then, the transfer function is

$$H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\mathrm{j}\omega)}{\widetilde{V}_{\text{in}}(\mathrm{j}\omega)} = \frac{R_2 + \frac{1}{\mathrm{j}\omega C}}{R_1 + R_2 + \frac{1}{\mathrm{j}\omega C}} = \frac{1 + \mathrm{j}\omega R_2 C}{1 + \mathrm{j}\omega C(R_1 + R_2)} \tag{29}$$

At low frequencies, we have

$$\lim_{\omega \to 0} |H(\mathrm{j}\omega)| = \lim_{\omega \to 0} \frac{\sqrt{1 + (\omega R_2 C)^2}}{\sqrt{1 + (\omega C(R_1 + R_2))^2}} = 1 \tag{30}$$

while at high frequencies, we have

$$\lim_{\omega\to\infty} |H(\mathrm{j}\omega)| = \lim_{\omega\to\infty} \frac{\sqrt{1+(\omega R_2 C)^2}}{\sqrt{1+(\omega C(R_1+R_2))^2}} \tag{31}$$

$$= \lim_{\omega\to\infty} \frac{\sqrt{\frac{1}{\omega^2}+(R_2 C)^2}}{\sqrt{\frac{1}{\omega^2}+(C(R_1+R_2))^2}} \tag{32}$$

$$= \frac{CR_2}{C(R_1+R_2)} = \frac{R_2}{R_1+R_2} \tag{33}$$

So at high frequencies, this circuit behaves like a regular voltage divider with just R_1 and R_2, as if the capacitor had vanished. This circuit is like a combination of a low-pass filter and a voltage divider: low frequency inputs are preserved, and high-frequency signals are diminished.

Now, we can create our Bode plots. We can write our transfer function in the following form:

$$H(\mathrm{j}\omega) = \frac{\omega_p}{\omega_z}\frac{\omega_z+\mathrm{j}\omega}{\omega_p+\mathrm{j}\omega} \tag{34}$$

where $\omega_z = \frac{1}{R_2 C} = \frac{1}{(10^3)(10^{-6})} = 10^3$ and $\omega_p = \frac{1}{(R_1+R_2)C} = \frac{1}{(10^4)(10^{-6})} = 10^2$.

This transfer function is more complicated as it does not fit the basic low pass or high pass filter models we are used to so this is where using poles and zeros is very helpful.

The magnitude plot will start off at $20\log_{10}(|H(\mathrm{j}0)|) = 20\log_{10}(1) = 0\,\mathrm{dB}$.

At $\omega_p = 10^2$, the slope will decrease to $-20\frac{\mathrm{dB}}{\mathrm{dec}}$.

At $\omega_z = 10^3$, the slope will increase to $0\frac{\mathrm{dB}}{\mathrm{dec}}$, at which point the magnitude should be $20\log_{10}(|H(\mathrm{j}\infty)|) = 20\log_{10}(\frac{R_2}{R_1+R_2}) = 20\log_{10}(\frac{1}{10}) = -20\,\mathrm{dB}$.

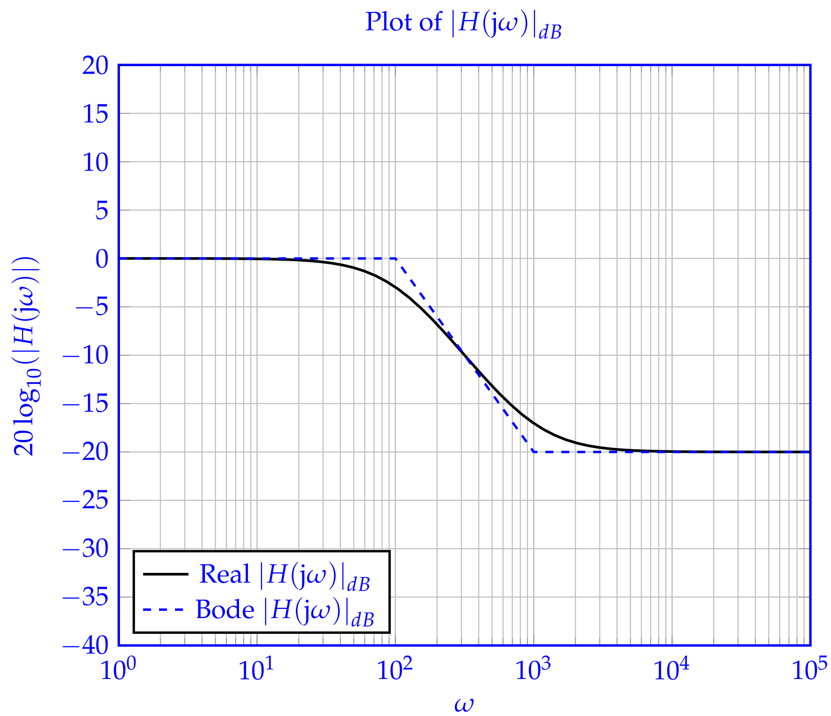
The phase plot will start off at 0.

From 10^1 to 10^2, the phase will decrease linearly in the Bode plot from 0 to $-\frac{\pi}{4}$ due to $\omega_p = 10^2$.

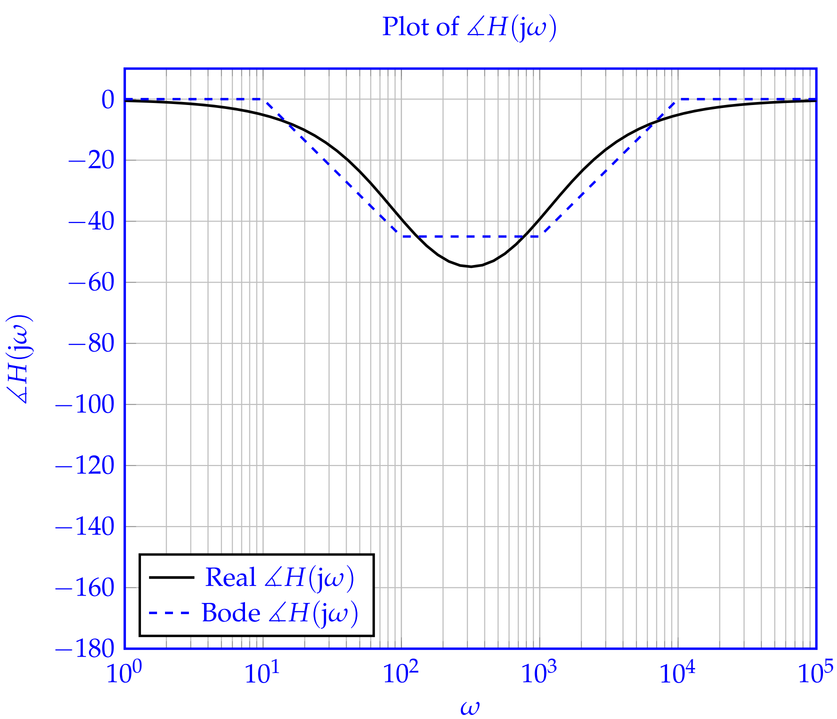
From 10^2 to 10^3, both the pole and zero change the phase and cancel each other's effects out so the phase stays constant at $-\frac{\pi}{4}$.

From 10^3 to 10^4, the phase will increase linearly in the Bode plot from $-\frac{\pi}{4}$ to 0 due to $\omega_z = 10^3$.

The magnitude and phase plots of $H(\mathrm{j}\omega)$ are shown below.



Plot of $|H(j\omega)|_{dB}$
$20\log_{10}(|H(j\omega)|)$
20
15
10
5
0
−5
−10
−15
−20
−25
−30
−35
−40
10^0
10^1
10^2
10^3
10^4
10^5
ω
Real $|H(j\omega)|_{dB}$
Bode $|H(j\omega)|_{dB}$



Plot of $\measuredangle H(j\omega)$
$\measuredangle H(j\omega)$
0
−20
−40
−60
−80
−100
−120
−140
−160
−180
10^0
10^1
10^2
10^3
10^4
10^5
ω
Real $\measuredangle H(j\omega)$
Bode $\measuredangle H(j\omega)$

(d) **Assuming $v_{in}(t) = 12\sin(\omega_{in}t)$ compute the $v_{out}(t)$ using the transfer function computed in part 1.a.** Remember that $R = 1\,\text{k}\Omega$ and $C = 1\,\mu\text{F}$ for this circuit, and assume $\omega_{in} = 1000\,\frac{\text{rad}}{\text{s}}$. In words, what is the effect of the transfer function in part 1.a on the magnitude and phase of the input signal?

Solution: To get $v_{out}(t)$, we must first convert $v_{in}(t)$ into phasor domain to get $\widetilde{V}_{in}(j\omega)$, then apply the transfer function to get $\widetilde{V}_{out}(j\omega)$, and then convert back to time domain to get $v_{out}(t)$.

To convert from time domain to phasor domain, we use the definition we derived in the previous discussion:

$$v_{in}(t) = V_0\cos(\omega t + \theta) \leftrightarrow \widetilde{V}_{in}(j\omega) = V_0 e^{j\theta} \tag{35}$$

Firstly, note that $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$, so we can write $v_{in} = 12\sin(\omega t)$ as $v_{in} = 12\cos\left(\omega t - \frac{\pi}{2}\right)$. Pattern matching with the phasor definition (with $V_0 = 12$ and $\phi = -\frac{\pi}{2}$),

$$\widetilde{V}_{in}(j\omega) = 12e^{-j\frac{\pi}{2}} \tag{36}$$

Now, we can find $\widetilde{V}_{out}(j\omega)$ by multiplying the transfer function with the output phasor. Note that we have to evaluate the transfer function at $\omega = \omega_{in} = 1000\,\frac{\text{rad}}{\text{s}}$ since that is the input angular frequency:

$$H(j\omega_{in}) = \frac{j(10^3)(10^3)(10^{-6})}{1 + j(10^3)(10^3)(10^{-6})} \tag{37}$$

$$= \frac{j}{1+j} \tag{38}$$

We will write $H(j\omega_{in})$ in the form $|H(j\omega_{in})|e^{j\angle H(j\omega_{in})}$, so that multiplying with $\widetilde{V}_{in}(j\omega)$ will be easier. First, to find $|H(j\omega_{in})|$:

$$|H(j\omega_{in})| = \left|\frac{j}{1+j}\right| = \frac{1}{\sqrt{2}} \tag{39}$$

Next, to find $\angle H(j\omega_{in})$:

$$\angle H(j\omega_{in}) = \angle(j) - \angle(1+j) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \tag{40}$$

Hence, $H(j\omega_{in}) = \frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}$, and

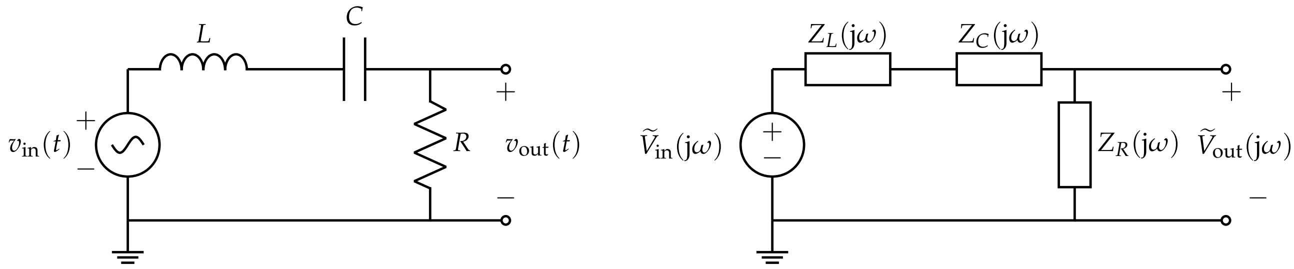
$$\widetilde{V}_{out}(j\omega_{in}) = H(j\omega_{in})\widetilde{V}_{in}(j\omega_{in}) = 6\sqrt{2}e^{-j\frac{\pi}{4}} \tag{41}$$

The last step is changing back to the time domain. For this step, we can use the phasor definition in the reverse direction:

$$v_{out}(t) = 6\sqrt{2}\cos\left(1000t - \frac{\pi}{4}\right) \tag{42}$$

2. (OPTIONAL) Band-Pass Filter

It is quite common to need to design a filter which selects only a narrow range of frequencies. One example is in WiFi radios, it is desirable to select only the 2.4GHz frequency containing your data, and reject information from other nearby cellular or bluetooth frequencies. This type of filter is called a band-pass filter; we will explore the design of this type of filter in this problem.



(a) Circuit in "time domain" **(b)** Circuit in "phasor domain"

(a) **Write down the transfer function** $H(j\omega) = \frac{\widetilde{V}_{out}(j\omega)}{\widetilde{V}_{in}(j\omega)}$ **for this circuit.**

Solution: Using the same voltage divider rule we've used in the past, $\widetilde{V}_{out}(j\omega)$ is:

$$\widetilde{V}_{out}(j\omega) = \widetilde{V}_{in}(j\omega)\frac{Z_R}{Z_R + Z_L + Z_C} \tag{43}$$

$$= \widetilde{V}_{in}\frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)} \tag{44}$$

$$\implies H(j\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} \tag{45}$$

$$= \frac{R}{R + j(\omega L - \frac{1}{\omega C})} \tag{46}$$

(b) Consider the inductor, capacitor, and resistor connected in series. **Write down the impedance of the series RLC combination in the form** $Z_{RLC}(j\omega) = A(\omega) + jX(\omega)$**, where** $A(\omega)$ **and** $X(\omega)$ **are real valued functions of** ω**. At what frequency** ω_r **does** $X(\omega_r) = 0$**?** (i.e. at what frequency is the impedance of the series combination of RLC purely real — meaning that the imaginary terms coming from the capacitor and inductor completely cancel each other. This is called the *resonant frequency*.)

Solution: Recall that the series impedance is the denominator of the voltage divider formula. From the previous part, $Z_{RLC} = Z_R + Z_L + Z_C = R + j(\omega L - \frac{1}{\omega C})$. Thus, $A(\omega) = R$ and $X(\omega) = \omega L - \frac{1}{\omega C}$.

Now, we can proceed to find ω_r.

$$X(\omega_r) = \omega_r L - \frac{1}{\omega_r C} = 0 \tag{47}$$

Multiplying both sides by ω_r:

$$\omega_r^2 L - \frac{1}{C} = 0 \tag{48}$$

$$\omega_r = \frac{1}{\sqrt{LC}}. \tag{49}$$

(c) **Find an expression for $|H(\mathrm{j}\omega)|$. What is $|H(\mathrm{j}\omega_r)|$? What is $|H(\mathrm{j}\omega_r/10)|$? What is $|H(\mathrm{j}10\omega_r)|$?** Rank the three quantities: $|H(\mathrm{j}\omega_r)|$, $|H(\mathrm{j}\omega_r/10)|$, $|H(\mathrm{j}10\omega_r)|$. What do you think the magnitude plot looks like?

Solution: We can compute $|H(\mathrm{j}\omega)|$ as follows:

$$|H(\mathrm{j}\omega)| = \left|\frac{R}{R+\mathrm{j}(\omega L - \frac{1}{\omega C})}\right| \tag{50}$$

$$= \frac{|R|}{\left|R+\mathrm{j}(\omega L - \frac{1}{\omega C})\right|} \tag{51}$$

$$= \frac{R}{\sqrt{\left(R+\mathrm{j}(\omega L - \frac{1}{\omega C})\right)\left(R-\mathrm{j}(\omega L - \frac{1}{\omega C})\right)}} \tag{52}$$

$$= \frac{R}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \tag{53}$$

Note that the only part that depends on ω is the $X(\omega) = \omega L - \frac{1}{\omega C}$ term in the denominator. At $\omega = \omega_r$, this term is 0. Hence,

$$|H(\mathrm{j}\omega_r)| = \frac{R}{\sqrt{R^2}} = 1 \tag{54}$$

At $\omega = \omega_r/10$, $X(\omega) = -9.9\sqrt{\frac{L}{C}}$. Similarly, at $\omega = 10\omega_r$, we have $X(\omega) = 9.9\sqrt{\frac{L}{C}}$. This means that $X(\omega_r/10)^2 = X(10\omega_r)^2$. Therefore,

$$|H(\mathrm{j}\omega_r/10)| = |H(\mathrm{j}10\omega_r)| = \frac{R}{\sqrt{R^2 + 9.9^2\frac{L}{C}}} < 1 \tag{55}$$

Thus, we would expect the graph of $|H(\mathrm{j}\omega)|$ to sharply peak at $\omega = \omega_r$ and decrease for $\omega > \omega_r$ and $\omega < \omega_r$. Specifically, $|H(\mathrm{j}\omega_r)| > |H(\mathrm{j}\omega_r/10)| = |H(\mathrm{j}10\omega_r)|$. This transfer function only "lets through" input signals with angular frequency $\omega = \omega_r$ and attenuates all other frequencies.

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