Discussion 6B

1. Plotting and Combining Transfer Functions

Recall that any transfer function (which is a complex function dependent on ω) can be written in polar form as

$$H(\mathrm{j}\omega) = |H(\mathrm{j}\omega)|\mathrm{e}^{\mathrm{j}\measuredangle H(\mathrm{j}\omega)} \tag{1}$$

where $|H(\mathrm{j}\omega)|$ and $\measuredangle H(\mathrm{j}\omega)$ are real functions of ω giving the magnitude and phase of the transfer function, respectively. To see how transfer functions combine, consider two transfer functions $H_1(\mathrm{j}\omega)$ and $H_2(\mathrm{j}\omega)$.

$$H_1(\mathrm{j}\omega) = |H_1(\mathrm{j}\omega)|\mathrm{e}^{\mathrm{j}\measuredangle H_1(\mathrm{j}\omega)} \tag{2}$$

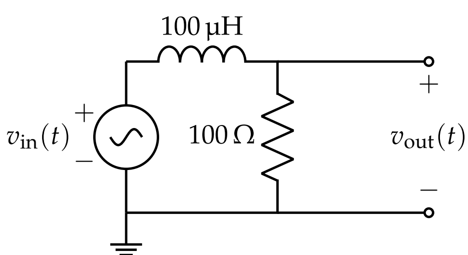
$$H_2(\mathrm{j}\omega) = |H_2(\mathrm{j}\omega)|\mathrm{e}^{\mathrm{j}\measuredangle H_2(\mathrm{j}\omega)} \tag{3}$$

$$H_1(\mathrm{j}\omega)\cdot H_2(\mathrm{j}\omega) = |H_1|\mathrm{e}^{\mathrm{j}\measuredangle H_1}|H_2|\mathrm{e}^{\mathrm{j}\measuredangle H_2} = |H_1||H_2|\mathrm{e}^{\mathrm{j}(\measuredangle H_1+\measuredangle H_2)} \tag{4}$$

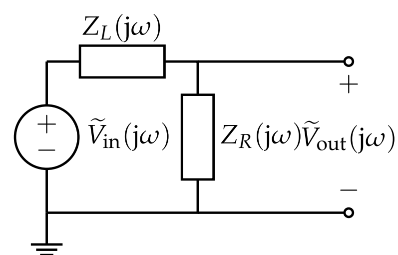
$$\frac{H_1(\mathrm{j}\omega)}{H_2(\mathrm{j}\omega)} = \frac{|H_1|\mathrm{e}^{\mathrm{j}\measuredangle H_1}}{|H_2|\mathrm{e}^{\mathrm{j}\measuredangle H_2}} = \frac{|H_1|}{|H_2|}\mathrm{e}^{\mathrm{j}(\measuredangle H_1-\measuredangle H_2)} \tag{5}$$

As you can see, magnitudes of transfer functions multiply/divide while the phases add/subtract.

In this problem we will examine the transfer function of fig. 1a.



(a) An LR filter in the "time-domain".



(b) An LR filter in the "phasor-domain".

Figure 1: Circuit schematic of LR filter in both domains.

(a) First, **solve for** $H(\mathrm{j}\omega)$. Then, **write expressions for** $|H(\mathrm{j}\omega)|$ **and** $\measuredangle H(\mathrm{j}\omega)$. For now, you can keep it in terms of R and L.

Solution: We use the voltage divider formula in the phasor domain:

$$\widetilde{V}_{out} = \frac{Z_R}{Z_R + Z_L}\widetilde{V}_{in}. \tag{6}$$

Substituting in the impedance formulas we know, we find that:

$$H(\mathrm{j}\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = \frac{R}{R+\mathrm{j}\omega L} = \frac{1}{1+\mathrm{j}\omega\frac{L}{R}}. \tag{7}$$

The magnitude can be found by dividing the magnitudes of the numerator and denominator:

$$|H(\mathrm{j}\omega)| = \frac{|1|}{\left|1+\mathrm{j}\omega\frac{L}{R}\right|} \tag{8}$$

$$= \frac{1}{\sqrt{1+\omega^2\frac{L^2}{R^2}}} \tag{9}$$

Similarly the phase can be found by subtracting the phase of the denominator from that of the numerator:

$$\measuredangle H(j\omega) = \measuredangle 1 - \measuredangle\left(1 + j\omega\frac{L}{R}\right) \tag{10}$$

$$= 0 - \text{atan2}\left(\omega\frac{L}{R}, 1\right) \tag{11}$$

(b) **What is the cutoff frequency for this circuit?**. Note that the values of the circuit elements are given in fig. 2a.

Recall that a transfer function of the form $H(j\omega) = \frac{k}{1+\frac{j\omega}{\omega_c}}$ is defined to have a cutoff frequency of ω_c. This is because a signal with an input frequency of ω_c exactly will have its magnitude attenuated by a factor of $\frac{1}{\sqrt{2}}$, which is a conventionally convenient number.

Solution: In this case, it will be the inverse of the LR time constant, that is

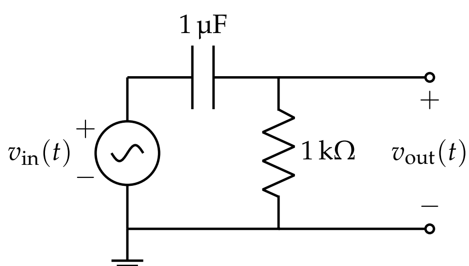
$$\omega_c = \frac{R}{L}. \tag{12}$$

For our given values, that's

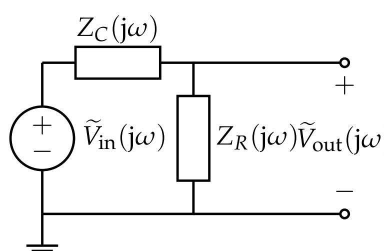
$$\omega_c = \frac{100\,\Omega}{100\,\mu\text{H}} = 1\times 10^6\,\frac{\text{rad}}{\text{s}}. \tag{13}$$

(c) Now suppose we want to compose the filter from fig. 2a with the filter from earlier (fig. 1a). Use $R = 1\,\text{k}\Omega$ and $C = 1\,\mu\text{F}$ for the RC filter. We can compose two circuits by connecting the output of the first circuit into the second circuit, through a unity gain buffer. For this problem, the transfer function of the LR filter from this worksheet fig. 1a is H_1, and the transfer function of the other RC filter is H_2. The transfer function of the composed circuit is:

$$H(j\omega) = H_1(j\omega)\cdot H_2(j\omega) \tag{14}$$



(a) An RC high-pass filter in the "time-domain".



(b) An RC high-pass filter in the "phasor-domain".

Draw this circuit.

(HINT: Follow the problem description. You will need to connect the output of the first filter to the input of the second filter in some way, but doing this directly would result in loading; you will need to add one more element to prevent this.)

Solution:

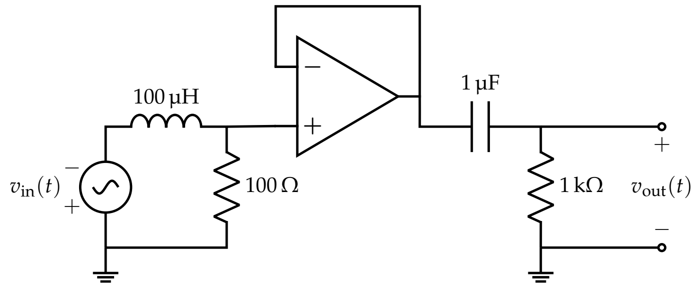


Figure 3: "Time-domain" circuit: Combination of the two filter circuits, through a buffer to avoid loading.

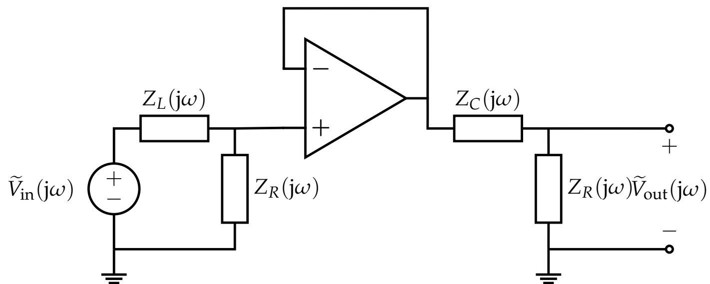


Figure 4: "Phasor-domain" circuit: Combination of the two filter circuits, through a buffer to avoid loading.

2. **Bode Plots (straight-line approximations) and filters**

Our eventual goal is to construct Bode plots of the following circuit, with $L = 100\,\mu\text{H}$, $C = 1\,\mu\text{F}$, $R_1 = 100\,\Omega$, and $R_2 = 1\,\text{k}\Omega$:

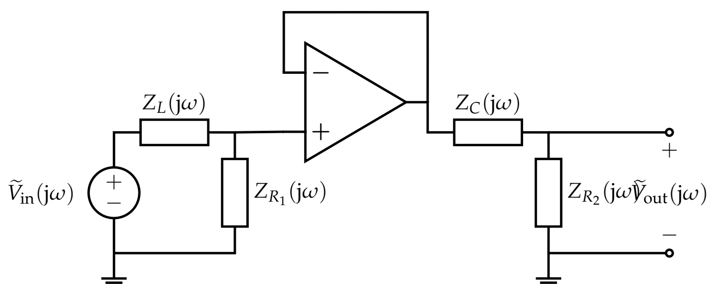


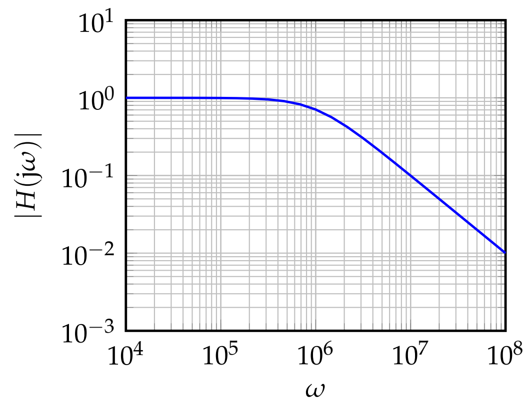
Figure 5

To do this we will leverage the fact that Bode plots can be composed in systematic ways.

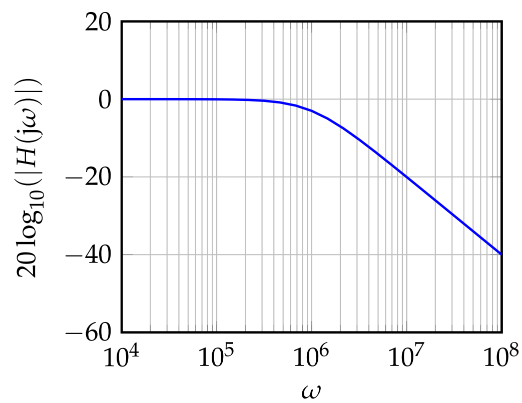
Before we dive into the problem, let's consider a modification of the *magnitude* plot that will help us work with multiple magnitude plots at once. Namely, instead of plotting $|H(j\omega)|$ vs. ω where the y-axis is on a *logarithmic* scale, we plot $20\log_{10}(|H(j\omega)|)$ vs. ω instead, and now the y-axis is on a *linear* scale. (This is known as decibel, or dB, scale.)

Why would we want to do this? Well, when combining magnitude transfer functions, we end up multiplying them. But we really want to add two plots *graphically* for simplicity, not multiply them, so we will just plot and add the logarithms. (The constant multiple 20 is there for convention reasons, related to decibels.)

Plot of $|H(j\omega)|$.

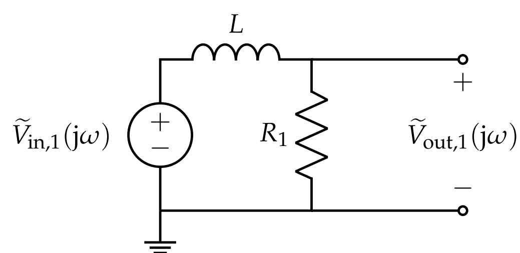


Linear $y-$axis: Plot of $|H(j\omega)|_{dB}$.



Notice that we do not need to do this for the *phase* plots, since their y axes are naturally in linear scale, and combining plots can already be done by addition. Now we are ready to begin working on the problems.

(a) Consider the first half of this circuit:



We learned in the previous problem that the transfer function is given by

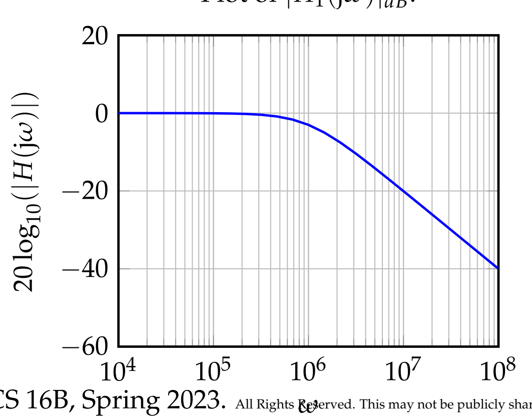
$$H_1(j\omega) = \frac{\widetilde{V}_{\text{out},1}}{\widetilde{V}_{\text{in},1}} = \frac{R_1}{R_1 + j\omega L} = \frac{1}{1 + j\omega \frac{L}{R_1}} \tag{15}$$

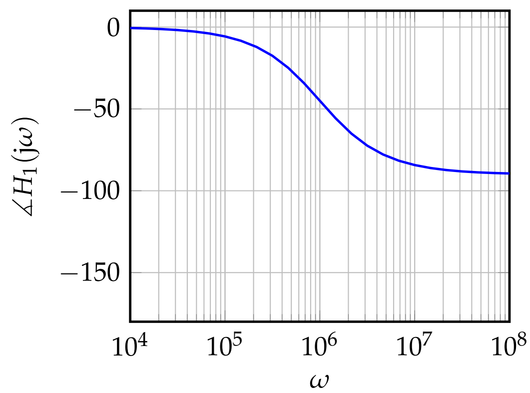
and the cutoff frequency $\omega_{c,1}$ is given by

$$\omega_{c,1} = \frac{R_1}{L} = \frac{100\,\Omega}{100\,\mu\text{H}} = 1 \times 10^6\,\frac{\text{rad}}{\text{s}} \tag{16}$$

If we plot $|H_1(j\omega)|_{dB}$ and $\measuredangle H_1(j\omega)$ using a computer, we would get the following:

Plot of $|H_1(j\omega)|_{dB}$.





On the above grids, **draw the Bode plots (piecewise linear approximations) for magnitude and phase**.

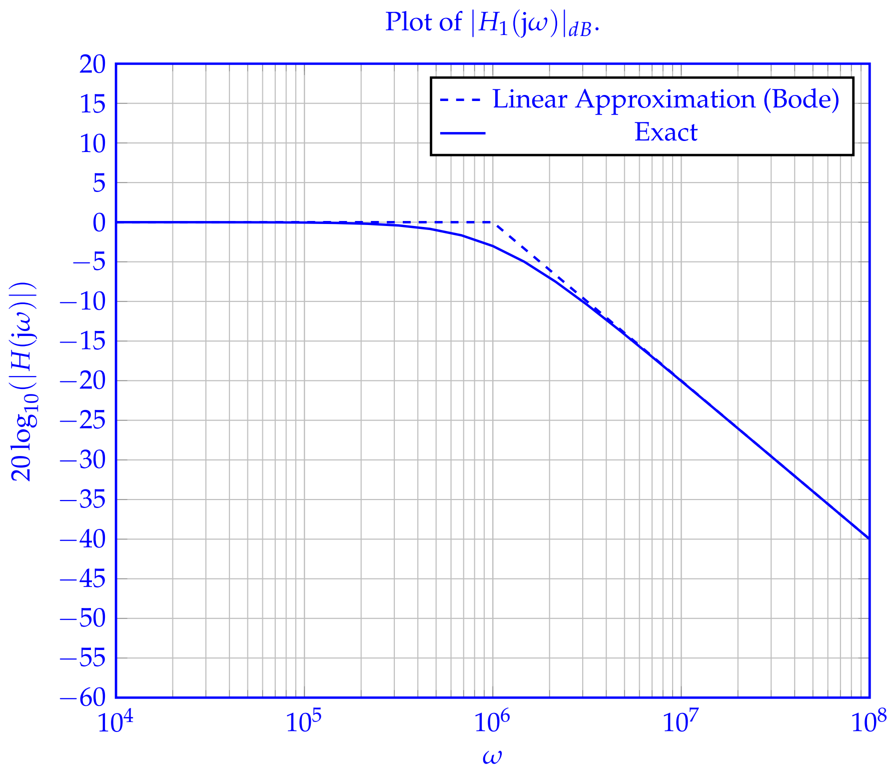
(HINT: Notice that both plots seem to have natural transitions that occur around the cutoff frequency $\omega_{c,1}$. For the magnitude plot, you should have two piecewise linear segments ($\omega < \omega_{c,1}$ and $\omega > \omega_{c,1}$). For the phase plot, you should have three piecewise linear segments ($\omega < \frac{1}{10}\omega_{c,1}$, $\frac{1}{10}\omega_{c,1} < \omega < 10\omega_{c,1}$, and $\omega > 10\omega_{c,1}$).)

Solution: One intuitive way to think about these plots is *graphically*, using aymptotes and approximating different curved segments as lines. However, there is also a much more mathematically-motivated approach to forming Bode Plots based the properties of logarithms (which is related to poles and zeros), which also explains when and why certain approximations are valid. For this analysis, see the Alternate Solution below.

Magnitude Bode Plot: Graphically, we notice that there are effectively 2 distinct regions of the plot to examine. At frequencies much below the cutoff $\omega \ll \omega_c$, the magnitude plot is effectively a horizontal line. So, we can draw that with a dashed segment. For frequencies much larger than cutoff $\omega \gg \omega_c$, we have a line with a decreasing slope (of -20 dB/decade). We similarly draw this asymptote, dashed.

Now, once we plot these both, there is a point of conflict in the middle, right around ω_c. In this region, we will effectively join the two models at a point, and pick the corresponding model for a given region (horizontal for $\omega < \omega_c$, sloped for $\omega > \omega_c$.)

You might wonder how we handle the fact that around ω_c, the sloped line claims that the magnitude at frequencies lower than ω_c should keep increasing, whereas the horizontal line in that region claims the magnitude is straight. Similarly, the horizontal line claims that the magnitude at frequencies higher than ω_c should stay constant, whereas the sloped line in that region claims the magnitude is decreasing. What we do here is default to unilaterally picking the model that is better for a given region. That's why we abruptly transition from one regime to the other; at $\omega_c - \epsilon$ for some small ϵ, the straight line is better, so we pick that curve. At $\omega_c + \epsilon$, we're now closer to the sloped model, so we start to slope down. This is to *maintain simplicity* while staying true (within bounded error) to the actual plot, which we know the shape of.

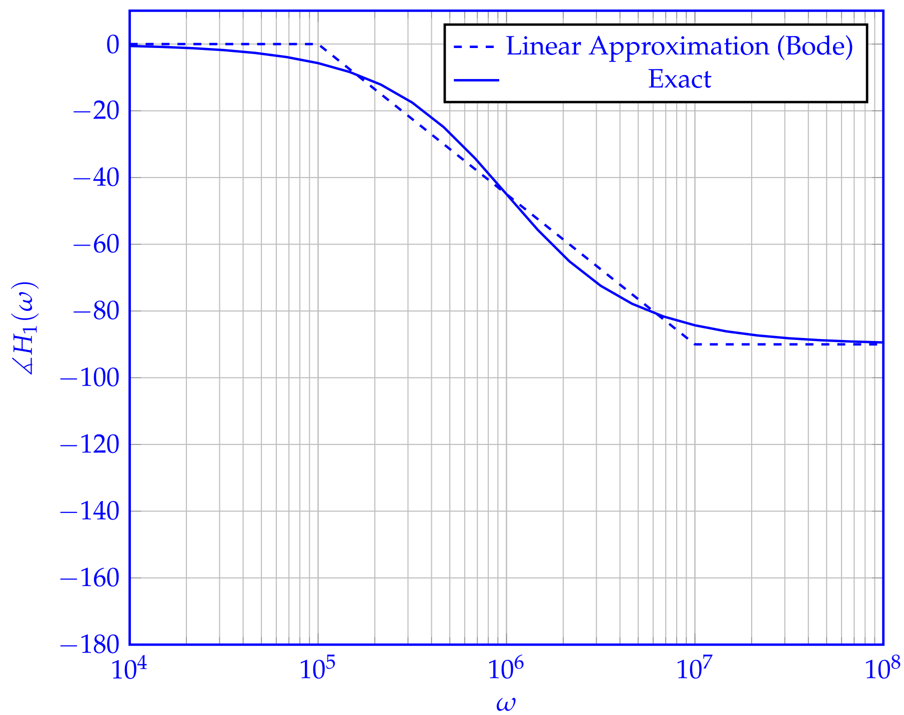


Phase Bode Plot: This one is a little bit trickier, since 2 line segments simply can't do a good job modeling the curvature of the actual phase. What's as simple as possible while being more detailed than 2 lines? 3 lines! So, we use 3 lines. The regimes we will follow are motivated by the natural division of the frequency axis into "decades", or factors of 10. So, we have 3 regions to examine:

- $\omega \le \frac{\omega_c}{10}$
- $\frac{\omega_c}{10} \le \omega \le 10\omega_c$
- $10\omega_c \le \omega$

In the first and third regions, where ω is significantly smaller than or larger than ω_c, we will approximate the cirves as horizontal lines. In the middle region, we join the other approximations by a straight line. We can show that the error with this approximation is bounded by about 6°, which is good enough for a first pass by hand when doing filter design.

Plot of $\measuredangle H_1(j\omega)$.



Alternate Solution (extends on the ideas of Poles and Zeros): We recognize that we can write $H_1(j\omega)$ in the form

$$H_1(j\omega) = \frac{1}{1 + j\omega \frac{L}{R_1}} = \frac{1}{1 + j\frac{\omega}{\omega_{c,1}}}. \tag{17}$$

Now we know the "recipe" to draw Bode plots, in particular

- For $\omega \ll \omega_{c,1}$,

$$H_1(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_{c,1}}} \approx \frac{1}{1} = 1. \tag{18}$$

  What this means is that

  - For the Bode plot of $|H_1(j\omega)|$ vs. ω:

$$20\log_{10}(|H_1(j\omega)|) \approx 20\log_{10}(1) = 0\,\text{dB}. \tag{19}$$

    Correspondingly, in the Bode plot, for $\omega < \omega_{c,1}$, the plot is constant with $20\log_{10}(|H_1(j\omega)|) = 0\,\text{dB}$.

  - For the Bode plot of $\measuredangle H_1(j\omega)$ vs. ω:

$$\measuredangle H_1(j\omega) \approx \measuredangle 1 = 0. \tag{20}$$

    Correspondingly, in the Bode plot, for $\omega < \omega_{c,1}/10$, the plot is constant with $\measuredangle H_1(j\omega) = 0$.

- For $\omega \gg \omega_{c,1}$,

$$H_1(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_{c,1}}} \approx \frac{1}{j\frac{\omega}{\omega_{c,1}}} = -j\frac{\omega_{c,1}}{\omega}. \tag{21}$$

  What this means is that

- For the Bode plot of $|H_1(j\omega)|$ vs. ω:

$$20\log_{10}(|H_1(j\omega)|) \approx 20\log_{10}\left(\frac{\omega_{c,1}}{\omega}\right) = 20\log_{10}(\omega_{c,1}) - 20\log_{10}(\omega). \tag{22}$$

Correspondingly, in the Bode plot, for $\omega > \omega_{c,1}$, the plot, starting at $(\omega_{c,1}, 0)$, decreases with slope -20 dB/decade.

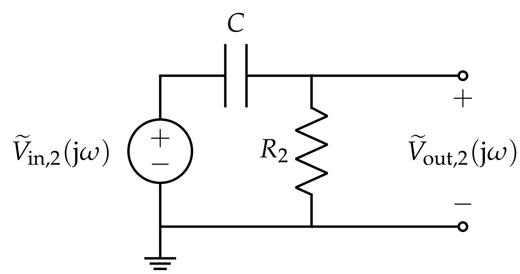
- For the Bode plot of $\measuredangle H_1(j\omega)$ vs. ω:

$$\measuredangle H_1(j\omega) \approx \measuredangle\left(-j\frac{\omega_{c,1}}{\omega}\right) = \measuredangle(-j) = -\frac{\pi}{2}. \tag{23}$$

Correspondingly, in the Bode plot, for $\omega > 10\omega_{c,1}$, the plot is constant with $\measuredangle H_1(\omega) = -\frac{\pi}{2}$.

- For ω such that $\omega_{c,1}/10 < \omega < 10\omega_{c,1}$, the behavior of the magnitude Bode plot is already defined, but not for the phase Bode plot. In this case we just define the plot to connect $(\omega_{c,1}/10, 0)$ and $(10\omega_{c,1}, -\frac{\pi}{2})$ by a line.

(b) Consider the second half of the circuit:



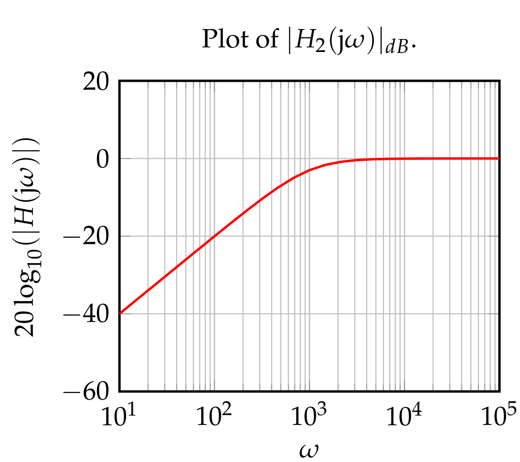
We learned in a previous discussion that the transfer function is given by

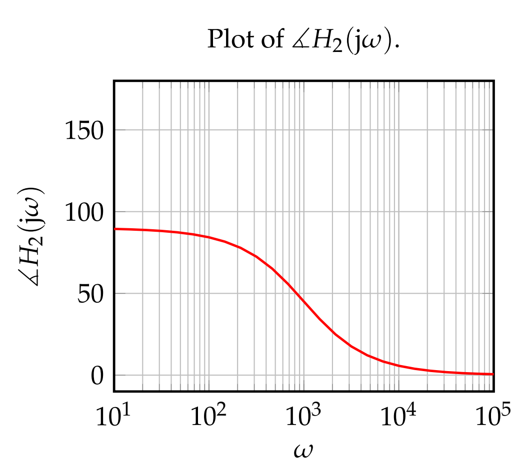
$$H_2(j\omega) = \frac{\widetilde{V}_{\text{out},2}}{\widetilde{V}_{\text{in},2}} = \frac{j\omega R_2 C}{1 + j\omega R_2 C} \tag{24}$$

and the cutoff frequency $\omega_{c,2}$ is given by

$$\omega_{c,2} = \frac{1}{R_2 C} = \frac{1}{(1\,\text{k}\Omega)\cdot(1\,\mu\text{F})} = 1 \times 10^3\ \frac{\text{rad}}{\text{s}} \tag{25}$$

If we plot $|H_2(j\omega)|_{dB}$ and $\measuredangle H_2(j\omega)$ using a computer, we would get the following:





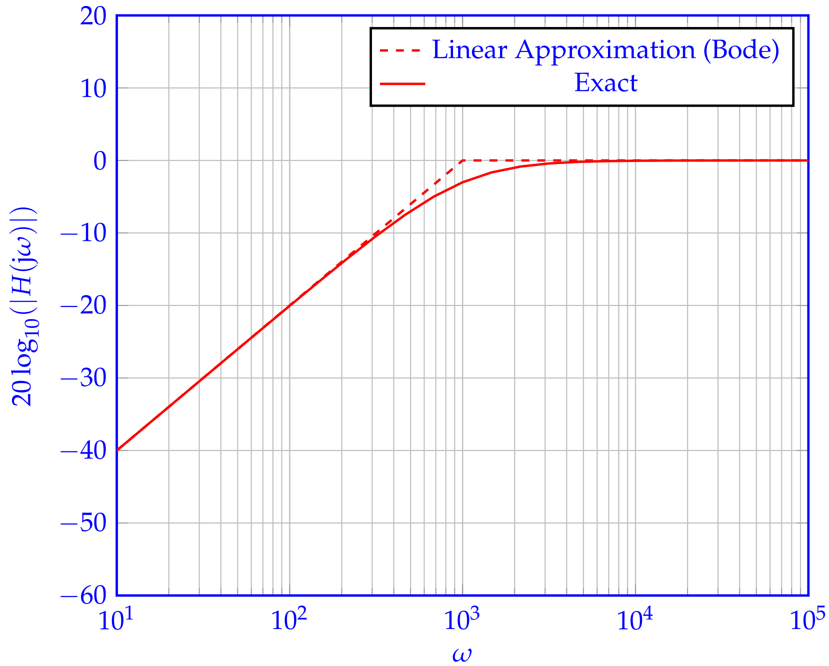
On the above grids, **draw the Bode plots (piecewise linear approximations) for magnitude and phase**.

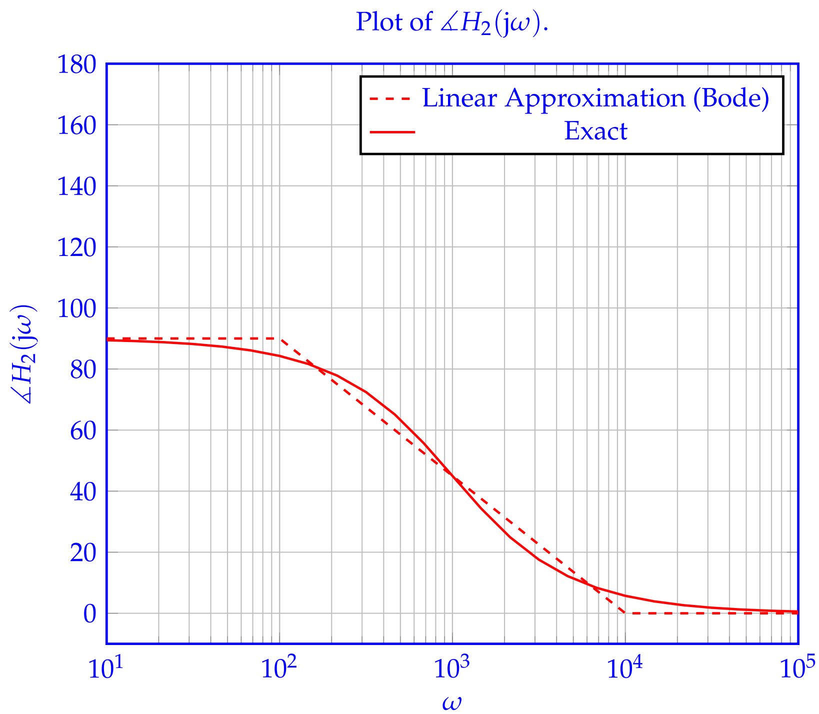
(HINT: Same hint as the previous part.)

Solution: As before, we will default to the graphical method here; for a more mathematical analysis using logarithms, see the Alternate Solution below.

These are highly similar to the previous subpart, but with the line segments "swapped" (for example for the Magnitude plot, sloped for smaller frequencies, and horizontal for larger frequencies). So, we don't need to repeat the analysis, and can draw the same lines as before in the new regions.

Plot of $|H_2(j\omega)|_{dB}$.





Alternate Solution (extends on the ideas of Poles and Zeros): We recognize that we can write $H_2(j\omega)$ in the form

$$H_2(j\omega) = \frac{j\omega R_2 C}{1 + j\omega R_2 C} = \frac{j\frac{\omega}{\omega_{c,2}}}{1 + j\frac{\omega}{\omega_{c,2}}}. \tag{26}$$

Now we know the "recipe" to draw Bode plots, in particular

- For $\omega \ll \omega_{c,2}$,

$$H_2(j\omega) = \frac{j\frac{\omega}{\omega_{c,2}}}{1 + j\frac{\omega}{\omega_{c,2}}} \approx \frac{j\frac{\omega}{\omega_{c,2}}}{1} = j\frac{\omega}{\omega_{c,2}}. \tag{27}$$

  What this means is that

  - For the Bode plot of $|H_2(j\omega)|$ vs. ω:

$$20\log_{10}(|H_2(j\omega)|) \approx 20\log_{10}\left(\frac{\omega}{\omega_{c,2}}\right) = 20\log_{10}(\omega) - 20\log_{10}(\omega_{c,2}). \tag{28}$$

    Correspondingly, in the Bode plot, for $\omega < \omega_{c,2}$, the plot increases with slope 20 dB/decade.

  - For the Bode plot of $\measuredangle H_2(\omega)$ vs. ω:

$$\measuredangle H_1(j\omega) \approx \measuredangle\left(j\frac{\omega}{\omega_{c,2}}\right) = \measuredangle j = \frac{\pi}{2}. \tag{29}$$

    Correspondingly, in the Bode plot, for $\omega < \omega_{c,2}/10$, the plot is constant with $\measuredangle H_1(j\omega) = \frac{\pi}{2}$.

- For $\omega \gg \omega_{c,2}$,

$$H_2(j\omega) = \frac{j\frac{\omega}{\omega_{c,2}}}{1 + j\frac{\omega}{\omega_{c,2}}} \approx \frac{j\frac{\omega}{\omega_{c,2}}}{j\frac{\omega}{\omega_{c,2}}} = 1. \tag{30}$$

  What this means is that

- For the Bode plot of $|H_2(j\omega)|$ vs. ω:

$$20\log_{10}(|H_2(j\omega)|) \approx 20\log_{10}(1) = 0\,\text{dB}. \tag{31}$$

Correspondingly, in the Bode plot, for $\omega > \omega_{c,2}$, the plot is constant with $20\log_{10}(|H_2(\omega)|) = 0\,\text{dB}$.

- For the Bode plot of $\measuredangle H_2(j\omega)$ vs. ω:

$$\measuredangle H_2(j\omega) \approx \measuredangle 1 = 0. \tag{32}$$

Correspondingly, in the Bode plot, for $\omega > 10\omega_{c,2}$, the plot is constant with $\measuredangle H_2(\omega) = 0$.

- For ω such that $\omega_{c,2}/10 < \omega < 10\omega_{c,2}$, the behavior of the magnitude Bode plot is already defined, but not for the phase Bode plot. In this case we just define the plot to connect $(\omega_{c,2}/10, \frac{\pi}{2})$ and $(10\omega_{c,2}, 0)$ by a line.

(c) Now, we will put this circuit together. Recall the diagram in fig. 5:

We saw earlier in the discussion that the transfer function is

$$H(j\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = H_1(j\omega)H_2(j\omega) = \frac{j\omega R_2 C}{(1 + j\omega\frac{L}{R_1})(1 + j\omega R_2 C)} \tag{33}$$

To plot $|H(j\omega)|_{dB}$ and $\measuredangle H(j\omega)$ we can use what we know about the plots for $H_1(j\omega)$ and $H_2(j\omega)$, as well as how the magnitude and phase of complex numbers change when multiplied with each other.

On the provided grids, **draw the Bode plots (piecewise linear approximations) for magnitude and phase**.

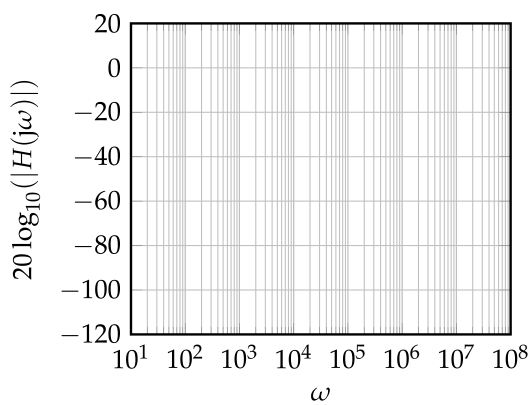
Hint: Recall that

$$20\log_{10}(|H(j\omega)|) = 20\log_{10}(|H_1(j\omega)H_2(j\omega)|) = 20\log_{10}(|H_1(j\omega)||H_2(j\omega)|) \tag{34}$$

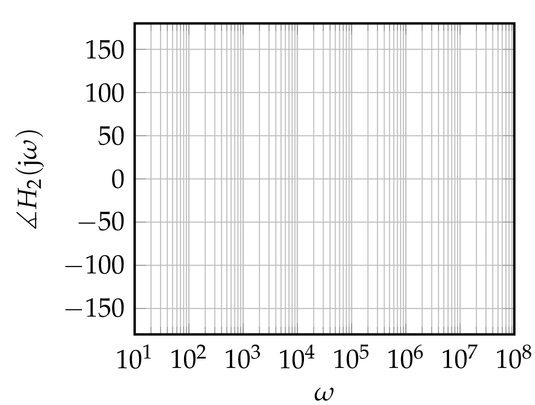
$$= 20\log_{10}(|H_1(j\omega)|) + 20\log_{10}(|H_2(\omega)|) \tag{35}$$

$$\measuredangle H(j\omega) = \measuredangle H_1(j\omega) + \measuredangle H_2(j\omega). \tag{36}$$

Plot of $|H_2(j\omega)|_{dB}$.

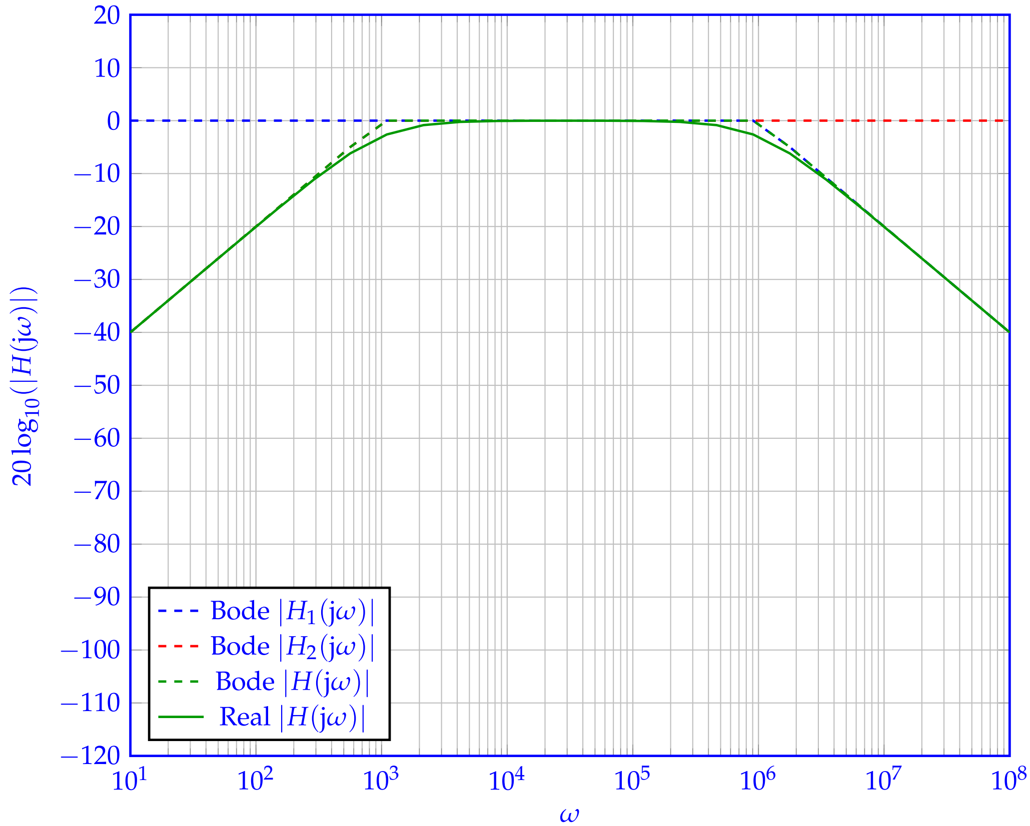


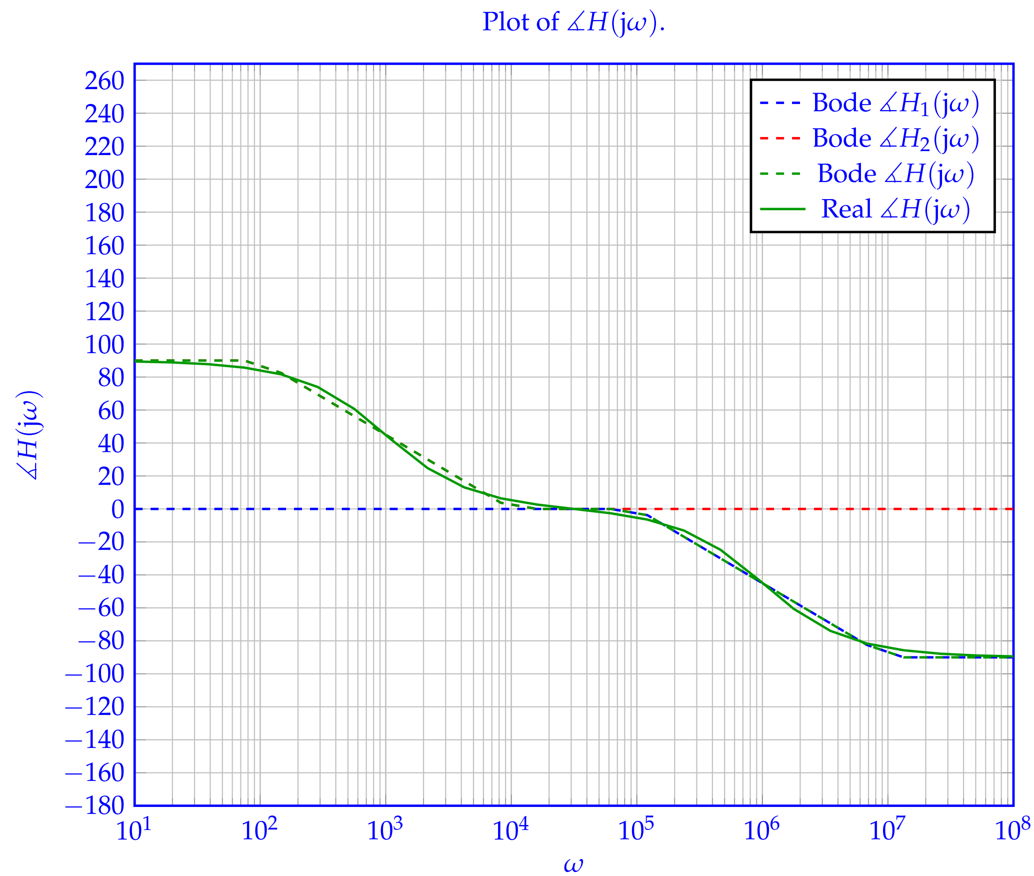
Plot of $\measuredangle H_2(j\omega)$.



Solution: What the hint means is that *we can add the plots (for both magnitude and phase) of $H_1(j\omega)$ and $H_2(j\omega)$ to get the plot for $H(j\omega)$.* In general this will let us do analysis of higher-order circuits by breaking them down into easily-analyzable chunks and adding the plots. Of course, since this property holds for the transfer functions, it holds for the Bode plots (which are good linear approximations to the transfer functions) too. So our plots end up looking like this:

Plot of $|H(j\omega)|_{dB}$.





Contributors:

- Neelesh Ramachandran.
- Alex Devonport.
- Nathan Lambert.
- Anant Sahai.
- Kareem Ahmad.
- Nikhil Jain.
- Druv Pai.