Discussion 7A

The following notes are useful for this discussion: Note 7.

1. Translating System of Differential Equations from Continuous Time to Discrete Time

Oftentimes, we wish to apply controls model on a computer. However, modeling a continuous time system on a computer is a nontrivial problem. Hence, we turn to discretizing our controls problem. That is, we define a discretized state $\vec{x}_d[i]$ and a discretized input $\vec{u}_d[i]$ that we "sample" every Δ seconds.

(a) Consider the scalar system below:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + bu(t). \tag{1}$$

where $x(t)$ is our state and $u(t)$ is our control input. Let $\lambda \neq 0$ be an arbitrary constant. Further suppose that our input $u(t)$ is piecewise constant, and that $x(t)$ is differentiable everywhere (and thus, continuous everywhere). That is, we define an interval $t \in [i\Delta, (i+1)\Delta)$ such that $u(t)$ is constant over this interval. Mathematically, we write this as

$$u(t) = u(i\Delta) = u_d[i] \text{ if } t \in [i\Delta, (i+1)\Delta). \tag{2}$$

The now-discretized input $u_d[i]$ is the same as the original input where we only "observe" a change in $u(t)$ every Δ seconds. Similarly, for $x(t)$,

$$x(t) = x(i\Delta) = x_d[i] \tag{3}$$

Let's revisit the solution for eq. (1), when we're given the initial conditions at t_0, i.e we know the value of $x(t_0)$ and want to solve for $x(t)$ at any time $t \geq t_0$:

$$x(t) = \mathrm{e}^{\lambda(t-t_0)}x(t_0) + b\int_{t_0}^{t} u(\theta)\mathrm{e}^{\lambda(t-\theta)}\,\mathrm{d}\theta \tag{4}$$

Given that we start at $t = i\Delta$, where $x(t) = x_d[i]$ is known, and satisfy eq. (1), where do we end up at $x_d[i+1]$? *(HINT): Think about the initial condition here. Where does our solution "start"?*

(b) Suppose we now have a continuous-time system of differential equations, that forms a vector differential equation. We express this with an input in vector form:

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = A\vec{x}(t) + \vec{b}u(t) \tag{5}$$

where $\vec{x}(t)$ is n-dimensional. Suppose further that the matrix A has distinct and non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. We collect the eigenvectors together and form the matrix $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$.

We now wish to find a matrix A_d and a vector $\vec{b}_d$ such that

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{6}$$

where $\vec{x}_d[i] = \vec{x}(i\Delta)$.

Firstly, define terms

$$\mathrm{e}^{\Lambda\Delta} = \begin{bmatrix} \mathrm{e}^{\lambda_1\Delta} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \mathrm{e}^{\lambda_n\Delta} \end{bmatrix} \tag{7}$$

$$\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \frac{1}{\lambda_n} \end{bmatrix} \tag{8}$$

$$\vec{\tilde{u}}_d[i] = V^{-1}\vec{b}u_d[i] \tag{9}$$

Note that the term $\mathrm{e}^{\Lambda\Delta}$ is just a label for our intents and purposes — this is not the same as applying e^x to every element in the matrix Λ.

Complete the following steps to derive a discretized system:

i. **Diagonalize the continuous time system using a change of variables (change of basis) to achieve a new system for $\vec{y}(t)$.**

ii. **Solve the diagonalized system. Remember that we only want a solution over the interval $t \in [i\Delta, (i+1)\Delta)$. Use the value at $t = i\Delta$ as your initial condition.**

iii. **Discretize the diagonalized system to obtain $\vec{y}_d[i]$. Show that**

$$\vec{y}_d[i+1] = \underbrace{\begin{bmatrix} \mathrm{e}^{\lambda_1\Delta} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \mathrm{e}^{\lambda_n\Delta} \end{bmatrix}}_{\mathrm{e}^{\Lambda\Delta}} \vec{y}_d[i] + \begin{bmatrix} \frac{\mathrm{e}^{\lambda_1\Delta}-1}{\lambda_1} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \frac{\mathrm{e}^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix} \vec{\tilde{u}}_d[i] \tag{10}$$

Then, show that the matrix $\begin{bmatrix} \frac{\mathrm{e}^{\lambda_1\Delta}-1}{\lambda_1} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \frac{\mathrm{e}^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix}$ **can be compactly written as $\Lambda^{-1}\left(\mathrm{e}^{\Lambda\Delta} - I\right)$.**

iv. **Undo the change of variables on the discretized diagonal system to get the discretized solution of the original system.**

(c) Consider the discrete-time system

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{11}$$

Suppose that $\vec{x}_d[0] = \vec{x}_0$. **Unroll the implicit recursion and show that the solution follows the form in eq. (12).**

$$\vec{x}_d[i] = A_d^i\vec{x}_d[0] + \left(\sum_{j=0}^{i-1} u_d[j]A_d^{i-1-j}\right)\vec{b}_d \tag{12}$$

You may want to verify that this guess works by checking the form of $\vec{x}_d[i+1]$. You don't need to worry about what A_d and $\vec{b}_d$ actually are in terms of the original parameters.

(Hint: If we have a scalar difference equation, how would you solve the recurrence? Try writing $\vec{x}_d[i]$ in terms of $\vec{x}_d[0]$ for $i = 1, 2, 3$ and look for a pattern.)

Contributors:

- Anish Muthali.
- Neelesh Ramachandran.
- Druv Pai.
- Anant Sahai.
- Nikhil Shinde.
- Sanjit Batra.
- Aditya Arun.
- Kuan-Yun Lee.
- Kumar Krishna Agrawal.

