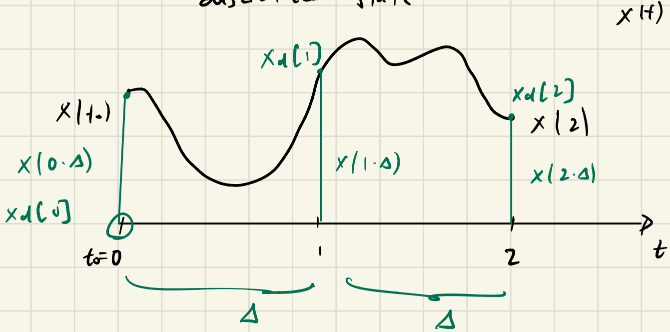
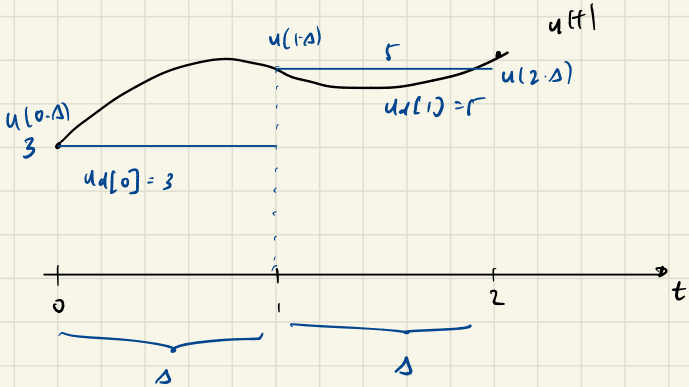
discretized state



$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$$

intervals $\Delta = 1$



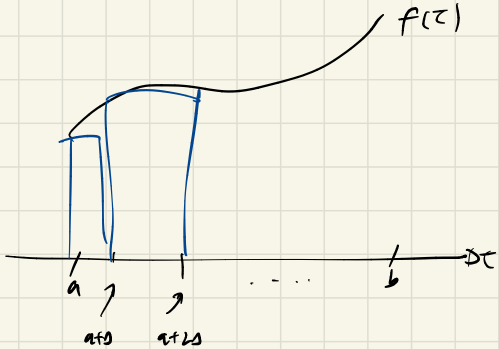
$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$$

$$x(t_0) = x_0$$

$$x(t) \quad \forall t \in [i\Delta, (i+1)\Delta)$$

$$x(t_0)$$
$$x(t_0+\Delta)$$

$$\int_a^b f(\tau)\, d\tau$$



Discussion 7A

The following notes are useful for this discussion: Note 9.

1. Translating System of Differential Equations from Continuous Time to Discrete Time

Oftentimes, we wish to apply controls model on a computer. However, modeling a continuous time system on a computer is a nontrivial problem. Hence, we turn to discretizing our controls problem. That is, we define a discretized state $\vec{x}_d[i]$ and a discretized input $\vec{u}_d[i]$ that we "sample" every Δ seconds.

(a) Consider the scalar system below:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + bu(t). \tag{1}$$

where $x(t)$ is our state and $u(t)$ is our control input. Let $\lambda \neq 0$ be an arbitrary constant. Further suppose that our input $u(t)$ is piecewise constant, and that $x(t)$ is differentiable everywhere (and thus, continuous everywhere). That is, we define an interval $t \in [i\Delta, (i+1)\Delta)$ such that $u(t)$ is constant over this interval. Mathematically, we write this as

$$u(t) = u(i\Delta) = u_d[i] \text{ if } t \in [i\Delta, (i+1)\Delta). \tag{2}$$

The now-discretized input $u_d[i]$ is the same as the original input where we only "observe" a change in $u(t)$ every Δ seconds. Similarly, for $x(t)$,

$$x(t) = x(i\Delta) = x_d[i] \tag{3}$$

Let's revisit the solution for eq. (1), when we're given the initial conditions at t_0, i.e we know the value of $x(t_0)$ and want to solve for $x(t)$ at any time $t \geq t_0$:

$$x(t) = \mathrm{e}^{\lambda(t-t_0)}x(t_0) + b\int_{t_0}^{t} u(\theta)\mathrm{e}^{\lambda(t-\theta)}\,\mathrm{d}\theta \tag{4}$$

Given that we start at $t = i\Delta$, where $x(t) = x_d[i]$ is known, and satisfy eq. (1), where do we end up at $x_d[i+1]$? *(HINT): Think about the initial condition here. Where does our solution "start"?*

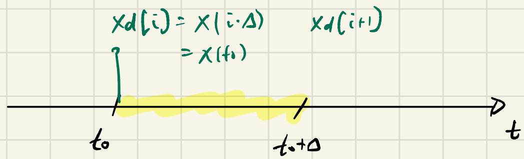
$t \in [i\Delta, (i+1)\Delta)$

start $t_0 = i\Delta$

$x(t_0) = x_d[i]$ is known

$u(t) = u_d[i]$

where do we end up at $x_d[i+1] = x((i+1)\cdot\Delta) = x(i\cdot\Delta + \Delta) = x(t_0+\Delta)$



$$x(t) = e^{\lambda(t-t_0)} x(t_0) + \underbrace{b\int_{t_0}^{t} u(\theta)\, e^{\lambda(t-\theta)}\, d\theta}_{b\int_{t_0}^{t} u(\theta)\, e^{\lambda t} e^{-\lambda\theta}\, d\theta}$$

$$x(t) = e^{\lambda(t-t_0)} x(t_0) + b\, e^{\lambda t}\int_{t_0}^{t} u(\theta)\, e^{-\lambda\theta}\, d\theta$$

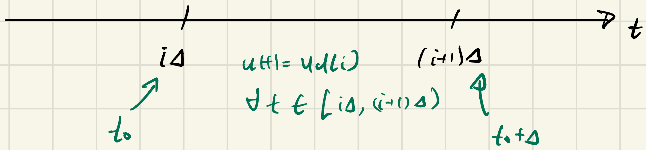
$$\begin{aligned} x_d[i+1] &= x((i+1)\cdot\Delta) \\ &= x(i\cdot\Delta + \Delta) \\ &= x(t_0+\Delta) \end{aligned}$$

$$x(t_0+\Delta) = e^{\lambda((t_0+\Delta)-t_0)} x(t_0) + b\, e^{\lambda(t_0+\Delta)}\int_{t_0}^{t_0+\Delta} u(\theta)\, e^{-\lambda\theta}\, d\theta$$

over $t \in [i\Delta, (i+1)\Delta)$

$u(t) = u_d[i]$

$$x(t_0+\Delta) = e^{\lambda((t_0+\Delta)-t_0)} x(t_0) + b\, e^{\lambda(t_0+\Delta)}\int_{t_0}^{t_0+\Delta} u_d[i]\, e^{-\lambda\theta}\, d\theta$$



$$x(t_0+\Delta) = e^{\lambda((t_0+\Delta)-t_0)} x(t_0) + b\, e^{\lambda(t_0+\Delta)} \int_{t_0}^{t_0+\Delta} u_d[i]\, e^{-\lambda\theta}\, d\theta$$

$$x(t_0+\Delta) = e^{\lambda\Delta} x(t_0) + b\, e^{\lambda(t_0+\Delta)} u_d[i] \underbrace{\int_{t_0}^{t_0+\Delta} e^{-\lambda\theta}\, d\theta}$$

$$\left[\frac{1}{-\lambda} e^{-\lambda\theta}\right]_{t_0}^{t_0+\Delta}$$

$$-\frac{1}{\lambda}\left[e^{-\lambda\theta}\right]_{t_0}^{t_0+\Delta}$$

$$\frac{1}{\lambda}\left[e^{-\lambda\theta}\right]_{t_0+\Delta}^{t_0}$$

$$\frac{1}{\lambda}\left[e^{-\lambda t_0} - e^{-\lambda(t_0+\Delta)}\right]$$

$$x(t_0+\Delta) = e^{\lambda\Delta} x(t_0) + b\, e^{\lambda(t_0+\Delta)} u_d[i] \cdot \frac{1}{\lambda}\left[e^{-\lambda t_0} - e^{-\lambda(t_0+\Delta)}\right]$$

$$+ \frac{b\, u_d[i]}{\lambda}\left[e^{\lambda\Delta} - 1\right]$$

$$e^{\lambda(t_0+\Delta)} e^{-\lambda t_0} = e^{(\lambda t_0 + \lambda\Delta) - \lambda t_0} = e^{\lambda\Delta}$$

$$e^{\lambda(t_0+\Delta)} e^{-\lambda(t_0+\Delta)} = e^{0} = 1$$

$$x(t_0+\Delta) = e^{\lambda\Delta} x(t_0) + b e^{\lambda(t_0+\Delta)} u_d[i] \cdot \frac{1}{\lambda}\left[e^{-\lambda t_0} - e^{-\lambda(t_0+\Delta)}\right]$$

$$+ \frac{b\, u_d[i]}{\lambda}\left[e^{\lambda\Delta} - 1\right]$$

$$e^{\lambda(t_0+\Delta)} e^{-\lambda t_0} = e^{(\lambda t_0+\lambda\Delta)-\lambda t_0} = e^{\lambda\Delta}$$

$$e^{\lambda(t_0+\Delta)} e^{-\lambda(t_0+\Delta)} = e^{0} = 1$$

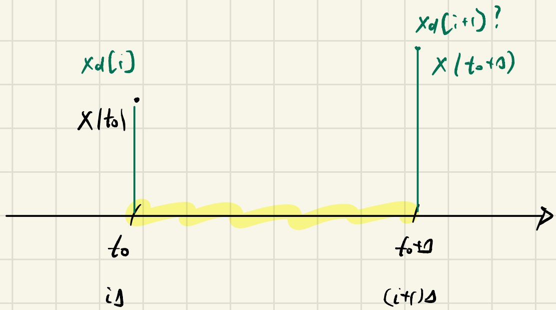
$$\overset{x_d[i+1]}{x(t_0+\Delta)} = e^{\lambda\Delta}\,\overset{x_d[i]}{x(t_0)} + b\, u_d[i]\left[\frac{e^{\lambda\Delta}-1}{\lambda}\right]$$

$x_d[i+1]$, $x_d[i]$

$$x_d[i+1] = x\big((i+1)\Delta\big) = x(i\Delta+\Delta) = x(t_0+\Delta)$$

$$x_d(i) = x(i\cdot\Delta) = x(t_0)$$

$$x_d[i+1] = e^{\lambda\Delta}\, x_d[i] + b\, u_d[i]\left[\frac{e^{\lambda\Delta}-1}{\lambda}\right]$$



DISCRETIZATION

(b) Suppose we now have a continuous-time system of differential equations, that forms a vector differential equation. We express this with an input in vector form:

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = A\vec{x}(t) + \vec{b}u(t) \tag{5}$$

where $\vec{x}(t)$ is n-dimensional. Suppose further that the matrix A has distinct and non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. We collect the eigenvectors together and form the matrix $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$.

We now wish to find a matrix A_d and a vector $\vec{b}_d$ such that

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{6}$$

where $\vec{x}_d[i] = \vec{x}(i\Delta)$.

Firstly, define terms

$$\mathrm{e}^{\Lambda\Delta} = \begin{bmatrix} \mathrm{e}^{\lambda_1\Delta} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \mathrm{e}^{\lambda_n\Delta} \end{bmatrix} \tag{7}$$

$$\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \frac{1}{\lambda_n} \end{bmatrix} \tag{8}$$

$$\vec{\tilde{u}}_d[i] = V^{-1}\vec{b}u_d[i] \tag{9}$$

Note that the term $\mathrm{e}^{\Lambda\Delta}$ is just a label for our intents and purposes — this is not the same as applying e^x to every element in the matrix Λ.

Complete the following steps to derive a discretized system:

i. **Diagonalize the continuous time system using a change of variables (change of basis) to achieve a new system for $\vec{y}(t)$.**

ii. **Solve the diagonalized system. Remember that we only want a solution over the interval $t \in [i\Delta, (i+1)\Delta)$. Use the value at $t = i\Delta$ as your initial condition.**

iii. **Discretize the diagonalized system to obtain $\vec{y}_d[i]$. Show that**

$$\vec{y}_d[i+1] = \underbrace{\begin{bmatrix} \mathrm{e}^{\lambda_1\Delta} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \mathrm{e}^{\lambda_n\Delta} \end{bmatrix}}_{\mathrm{e}^{\Lambda\Delta}} \vec{y}_d[i] + \begin{bmatrix} \frac{\mathrm{e}^{\lambda_1\Delta}-1}{\lambda_1} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \frac{\mathrm{e}^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix} \vec{\tilde{u}}_d[i] \tag{10}$$

Then, show that the matrix $\begin{bmatrix} \frac{\mathrm{e}^{\lambda_1\Delta}-1}{\lambda_1} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \frac{\mathrm{e}^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix}$ **can be compactly written as** $\Lambda^{-1}\left(\mathrm{e}^{\Lambda\Delta} - I\right)$**.**

$$\frac{dx(t)}{dt} = Ax(t) + b\,u(t)$$

$$x = Vy$$

$$\frac{d}{dt} Vy = AVy + bu(t)$$

$$V \frac{d}{dt} y = AVy + bu(t)$$

$$\frac{d}{dt} y = \underbrace{V^{-1}AV}_{\Lambda} y + V^{-1}bu(t)$$

$$\frac{dy}{dt} = \Lambda y + \underbrace{V^{-1}bu(t)}_{V^{-1}b\,u_d[i]}$$

$\tilde{u_d}[i]$

only care about $t \in [i\Delta, (i+1)\Delta)$

assume $u(t) = u_d(i)$ $\forall t \in [i\Delta, (i+1)\Delta)$

$$\frac{dy}{dt} = \Lambda y + \tilde{u_d}[i]$$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} \tilde{u}_{d_1}[i] \\ \vdots \\ \tilde{u}_{d_n}[i] \end{bmatrix}$$

$$\frac{d}{dt} y_j = \lambda_j y_j + \tilde{u}_{d_j}[i]$$

SLEEPY

only care about $t \in [i\Delta, (i+1)\Delta)$

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we know $y_j(i\Delta)$

assume $\tilde{u}_{d_j}[i]$ is constant over this interval

$$y_{d_j}[i+1] = e^{\lambda_j \Delta} y_{d_j}[i] + \frac{e^{\lambda_j \Delta} - 1}{\lambda_j} \tilde{u}_{d_j}[i]$$

$y_d \longrightarrow x_d$ $\qquad x_d[i+1] = V y_d[i+1]$

VDE $\xrightarrow{\text{diagonalization}}$

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) \qquad \frac{d\vec{y}}{dt} = D\vec{y} \qquad \frac{d}{dt}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

n rows

coupled

n rows

decoupled

$$\frac{dy_i}{dt} = \lambda_i y_i$$

scalar

$\vec{x}(t)$

iv. **Undo the change of variables on the discretized diagonal system to get the discretized solution of the original system.**

(c) Consider the discrete-time system

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{11}$$

Suppose that $\vec{x}_d[0] = \vec{x}_0$. **Unroll the implicit recursion and show that the solution follows the form in eq. (12).**

$$\vec{x}_d[i] = A_d^i\vec{x}_d[0] + \left(\sum_{j=0}^{i-1} u_d[j] A_d^{i-1-j}\right)\vec{b}_d \tag{12}$$

You may want to verify that this guess works by checking the form of $\vec{x}_d[i+1]$. You don't need to worry about what A_d and $\vec{b}_d$ actually are in terms of the original parameters.

(Hint: If we have a scalar difference equation, how would you solve the recurrence? Try writing $\vec{x}_d[i]$ in terms of $\vec{x}_d[0]$ for $i = 1, 2, 3$ and look for a pattern.)

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