Noise

4.99 V

4.995 V

5.01 V

⋮

1. **System Identification by Means of Least Squares**

   (a) Consider the scalar discrete-time system

system parameters

State evolution

$$x[i+1] = ax[i] + bu[i] + w[i] \tag{1}$$

Where the scalar state at timestep i is $x[i]$, the input applied at timestep i is $u[i]$ and $w[i]$ represents some (small) external disturbance that also participated at timestep i (which we cannot predict or control, it's a purely random disturbance).

Assume that you have measurements for the states $x[i]$ from $i = 0$ to ℓ and also measurements for the controls $u[i]$ from $i = 0$ to $\ell - 1$. Further assume $\ell \geq 2$.

Show that we can set up a linear system as in eq. (2) to find constants a and b. How do we solve this system?

$$\underbrace{\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[\ell] \end{bmatrix}}_{\vec{s}} \approx \underbrace{\begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \vdots \\ x[\ell-1] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{p}} \tag{2}$$

$\cancel{(x[-1], u[-1]) \rightarrow x[0]}$

$(x[0], u[0]) \rightarrow x[1]$ $\rightarrow$ $ax[0] + bu[0] = x[1]$

$(x[1], u[1]) \rightarrow x[2]$ $\quad$ $ax[1] + bu[1] = x[2]$

$\vdots$ $\quad$ $\vdots$

$(x[\ell-1], u[\ell-1]) \rightarrow x[\ell]$ $\quad$ $ax[\ell-1] + bu[\ell-1] = x[\ell]$

a^*, b^*

"closeness?"

"residual"

$$\left[x[1]-(ax[0]+bu[0])\right]^2$$

$$x[1]=ax[0]+bu[0]$$

$$x[1]-(ax[0]+bu[0])\approx 0$$

$$\vdots$$

$$\left[x[\ell]-(ax[\ell-1]+bu[\ell-1])\right]^2$$

$$\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[\ell] \end{bmatrix} - \begin{bmatrix} x[0] & u[0] \\ \vdots & \vdots \\ x[\ell-1] & u[\ell-1] \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x[1]-(ax[0]+bu[0]) \\ \vdots \\ x[\ell]-(ax[\ell-1]+bu[\ell-1]) \end{bmatrix}$$

$$\left\| \underbrace{\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[\ell] \end{bmatrix}}_{s} - \underbrace{\begin{bmatrix} x[0] & u[0] \\ \vdots & \vdots \\ x[\ell-1] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{p}} \right\|^2$$

$$\arg\min_{\vec{p}} \|s - D\vec{p}\|^2$$

warm up

find $\min_{x} (x^2+1)$

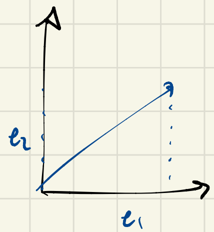
Least Squares

sum of squared errors

$$\left\| \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right\| = \sqrt{e_1^2 + e_2^2}$$

$$\left\| \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right\|^2 = \sqrt{e_1^2 + e_2^2}^{\,2} = e_1^2 + e_2^2$$

sum of squared errors



$$\arg\min_{\vec{p}} \| s - D\vec{p} \|^2$$

$$p^* = (D^T D)^{-1} D^T s$$

$D^T D$ might not be invertible!

(b) What if there were now two distinct scalar inputs to a scalar system

$$x[i+1] = ax[i] + b_1u_1[i] + b_2u_2[i] + w[i] \tag{3}$$

and that we have measurements as before, but now also for both of the control inputs.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a, b_1, b_2**.**

$$\left.\begin{aligned} x[1] &= a x[0] + b_1 u_1[0] + b_2 u_2[0] \\ &\vdots \\ x[\ell] &= a x[\ell-1] + b_1 u_1[\ell-1] + b_2 u_2[\ell-1] \end{aligned}\right\} \Rightarrow \underbrace{\begin{bmatrix} x[1] \\ \vdots \\ x[\ell] \end{bmatrix}}_{\vec{S}} = \underbrace{\begin{bmatrix} x[0] & u_1[0] & u_2[0] \\ \vdots & \vdots & \vdots \\ x[\ell-1] & u_1[\ell-1] & u_2[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b_1 \\ b_2 \end{bmatrix}}_{\vec{p}}$$

$$\vec{p}^* = (D^TD)^{-1}D^T\vec{S}$$

(c) **What could go wrong in the previous case? For what kind of inputs would make least-squares fail to give you the parameters you want?**

(d) Now consider the two dimensional state case with a single input.

$$\begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix}$$

$$\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \vec{x}[i] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i] + \vec{w}[i] \tag{4}$$

How can we treat this like two parallel problems to set this up using least-squares to get estimates for the unknown parameters $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$**?** Write the least squares solution in terms of your known matrices and vectors (including based on the labels you gave to various matrices/vectors in previous parts). *Hint: What work/computation can we reuse across the two problems?*

$$\left.\begin{aligned} x_1[1] &= a_{11} x_1[0] + a_{12} x_2[0] + b_1 u[0] \\ x_2[1] &= a_{21} x_1[0] + a_{22} x_2[0] + b_2 u[0] \\ x_1[2] &= a_{11} x_1[1] + a_{12} x_2[1] + b_1 u[1] \\ x_2[2] &= a_{21} x_1[1] + a_{22} x_2[1] + b_2 u[1] \end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1[1] \\ x_1[2] \\ \vdots \\ x_1[\ell] \end{bmatrix} = \begin{bmatrix} x_1[0] & x_2[0] & u[0] & 0 & 0 & 0 \\ & & & & & \\ & & & & & \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ b_1 \\ a_{21} \\ a_{22} \\ b_2 \end{bmatrix}$$

SCHNITZEL

Contributors:

- Anish Muthali.
- Neelesh Ramachandran.
- Anant Sahai.
- Regina Eckert.
- Kareem Ahmad.

