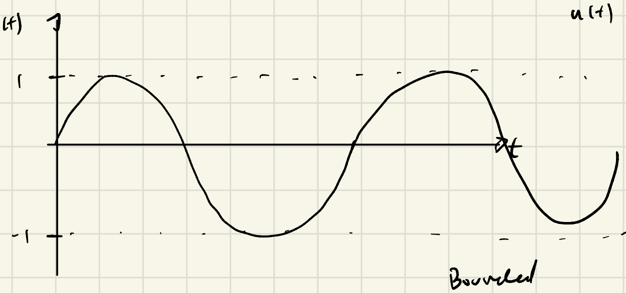
Stability

input and outputs being stable (bounded)

$u(t) = \sin(t) \in [-1, 1]$

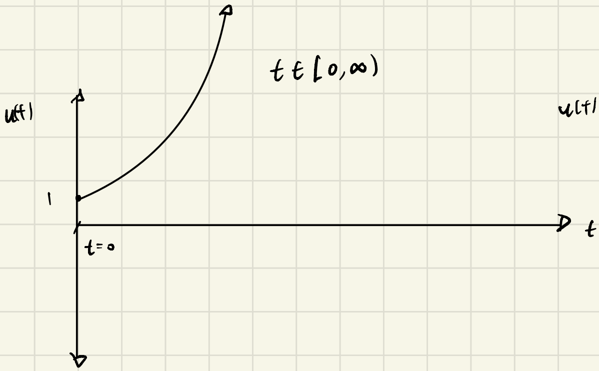
$|u(t)| \leq 1$

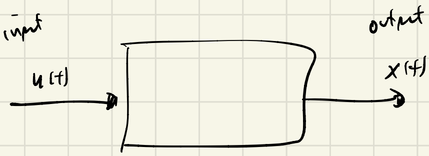


Bounded

$t \in [0, \infty)$

$u(t) = e^t$





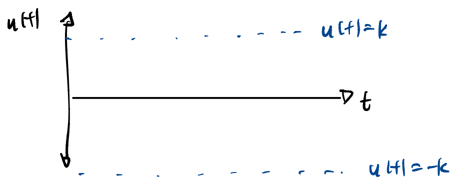
any bounded input produces a bounded output

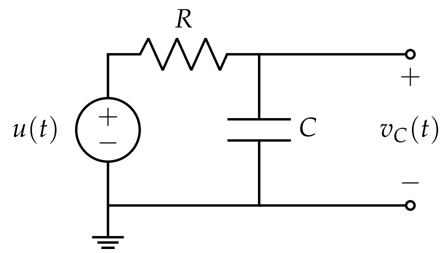
EECS 16B Designing Information Systems and Devices II UC Berkeley Spring 2023

Discussion 8B

1. Stability Examples and Counterexamples

(a) Consider the circuit below with $R = 1\,\Omega$, $C = 0.5\,\text{F}$, and $u(t)$ is some function bounded between $-K$ and K for some constant $K \in \mathbb{R}$ (for example $K\cos(t)$). Furthermore assume that $v_C(0) = 0\,\text{V}$ (that the capacitor is initially discharged).





BIBO stable

This circuit can be modeled by the differential equation

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = -2v_C(t) + 2u(t) \tag{1}$$

Show that the differential equation is always stable (that is, as long as the input $u(t)$ is bounded, $v_C(t)$ also stays bounded). Consider what this means in the physical circuit. *HINT: You may want to use the triangle inequality, i.e. $|a+b| \leq |a| + |b|$, and the triangle inequality for integrals, i.e. $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$. When we use $|\cdot|$ notation here, we will take this to mean the magnitude, rather than the absolute value (since we can be dealing with complex numbers).*

find $v_c(t)$

find a bound α

s.t. $|v_c(t)| \leq \alpha \quad \forall t$

$|a+b| \leq |a| + |b|$

$|a+(b+c)| \leq |a| + |b+c| \leq |a| + |b| + |c|$

$|b+c| \leq |b| + |c|$

$|a+b+c| \leq |a| + |b| + |c|$

Δ inequality

$|a_1 + a_2 + a_3 + \dots + a_N| = \left|\sum_{i=1}^{N} a_i\right| \leq \sum_{i=1}^{N} |a_i|$

$\left|\int_0^t f(\tau)\,d\tau\right| \leq \int_0^t |f(\tau)|\,d\tau$

Δ inequality

$$\frac{d v_c(t)}{dt} = \lambda v_c(t) + b u(t)$$

$$v_c(t) = v_c(0) e^{\lambda t} + b \int_0^t e^{\lambda(t-\tau)} u(\tau) d\tau$$

$$v_c(t) = v_c(0) e^{-2t} + 2 \int_0^t e^{-2(t-\tau)} u(\tau) d\tau$$

$$|v_c(t)| = \left| v_c(0) e^{-2t} + 2 \int_0^t e^{-2(t-\tau)} u(\tau) d\tau \right|$$

$$\leq \underbrace{\left| v_c(0) e^{-2t} \right|}_{\to 0} + \left| 2 \int_0^t e^{-2(t-\tau)} u(\tau) d\tau \right|$$

$$= \left| 2 \int_0^t e^{-2t} e^{2\tau} u(\tau) d\tau \right|$$

$$= \left| 2 e^{-2t} \int_0^t e^{2\tau} u(\tau) d\tau \right|$$

$$= \left| 2e^{-2t} \right| \left| \int_0^t e^{2\tau} u(\tau) d\tau \right| \qquad |a \cdot b| = |a||b|$$

$$\leq \left| 2e^{-2t} \right| \int_0^t \left| e^{2\tau} u(\tau) \right| d\tau$$

$$= \left| 2e^{-2t} \right| \int_0^t \left| e^{2\tau} \right| |u(\tau)| d\tau \qquad |u(\tau)| \leq k \quad \forall \tau$$

$$\leq \left| 2e^{-2t} \right| \int_0^t \left| e^{2\tau} \right| k \, d\tau$$

$$= 2k e^{-2t} \underbrace{\int_0^t e^{2\tau} d\tau}$$

$$\left[\frac{1}{2} e^{2\tau} \right]_0^t = \frac{1}{2} \left[e^{2t} - e^0 \right]$$

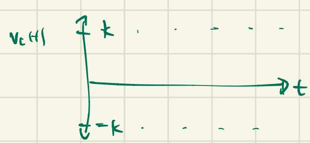
$$= \frac{1}{2} \left[e^{2t} - 1 \right]$$

$$= 2k e^{-2t} \left[\frac{1}{2} \cdot (e^{2t} - 1) \right]$$

$$= k \left[1 - e^{-2t} \right]$$

$$\leq k$$

$$|v_c(t)| \leq k \quad \forall t$$



(b) **(PRACTICE)** Now, suppose that in the circuit of part 1.a we replaced the resistor with an inductor as in fig. 1.

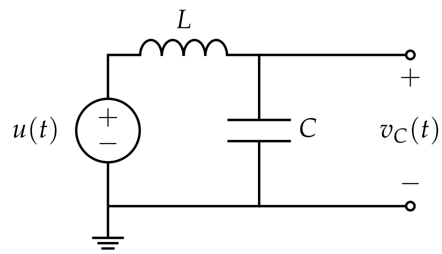


Figure 1: The original circuit with an inductor in place of the resistor.

Let $L = 1\,\text{mH}$. **Repeat part 1.a for the new circuit (with an inductor). Consider the following process to arrive at the result:**

i. **Derive the system of differential equations using KCL, KVL, and NVA. Show that the system is** $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$ **with the initial condition being** $\begin{bmatrix} v_C(0) \\ i_L(0) \end{bmatrix} = \vec{0}$**.**

ii. **Solve the matrix differential equation, using diagonalization if needed. Show that the diagonalized system has a solution**

$$\vec{y}(t) = \begin{bmatrix} \frac{1}{2LC} e^{\mathrm{j}\frac{1}{\sqrt{LC}}t} \int_0^t e^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta} u(\theta)\,\mathrm{d}\theta \\ \frac{1}{2LC} e^{-\mathrm{j}\frac{1}{\sqrt{LC}}t} \int_0^t e^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta} u(\theta)\,\mathrm{d}\theta \end{bmatrix} \tag{2}$$

where $\vec{y}(t) = V^{-1}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$ **for change of basis matrix** V**.** You may use the fact that the eigenvalue, eigenvector pairs of $\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}$ are $\left(\mathrm{j}\frac{1}{\sqrt{LC}}, \begin{bmatrix} -\mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}\right)$ and $\left(-\mathrm{j}\frac{1}{\sqrt{LC}}, \begin{bmatrix} \mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}\right)$.

iii. **Apply a similar process from part 1.a to show that, if we have a bounded input** $u(t)$**, then the system can grow unboundedly.** When showing that a system is unstable, it suffices to choose a bounded $u(t)$ that makes the system unbounded. We can choose $u(t) = 2\cos\left(\frac{1}{\sqrt{LC}}\right) = e^{\mathrm{j}\frac{1}{\sqrt{LC}}t} + e^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}$ [1]. *HINT: You may use the fact that* $i_L(t) = y_1(t) + y_2(t)$.

Hint: You might find it useful to revisit the process of generating the state-space equations for $v_C(t)$ *and* $i_L(t)$ *as done in Note 4 for the LC Tank. The difference is that here, we have an input voltage.*

[1] The natural frequency of this system is $\omega_n = \frac{1}{\sqrt{LC}}$. If we excite this system at a period equal to the natural frequency, we can make it grow unboundedly. This is similar to pushing a swing at the same rate it swings, which makes it swing farther.

(c) Thus far, we have dealt with continuous systems so it also makes sense to consider discrete systems. Consider the discrete system

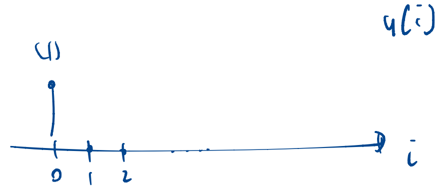
not BIBO stable

$$x[i+1] = 2x[i] + u[i] \tag{3}$$

with $x[0] = 0$.

Is the system stable or unstable? If unstable, find a bounded input sequence $u[i]$ that causes the system to "blow up".

$|u[i]| \le 1$



$x[1] = 2x[0] + u[0]$

$x[1] = 0 + 1$

$x[2] = 2x[1] + u[1]$

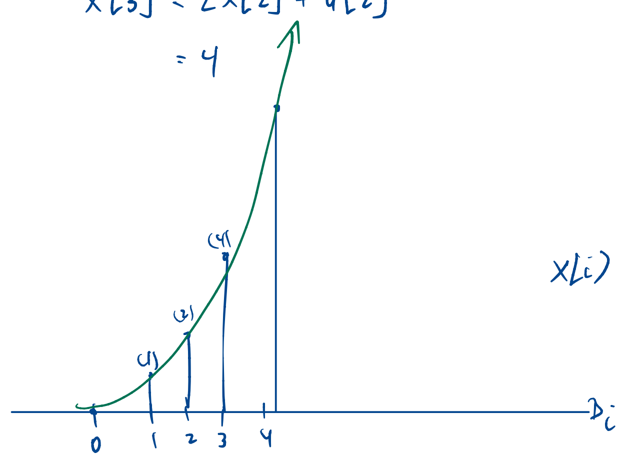
$= 2 + 0$

$= 2$

COUNTER EXAMPLE

$x[3] = 2x[2] + u[2]$

$= 4$



2. Changing behavior through feedback

In this question, we discuss how *feedback control* can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[i+1] = 0.9x[i] + u[i] + w[i] \tag{4}$$

where $u[i]$ is the control input we get to apply based on the current state and $w[i]$ is the external disturbance, each at time i.

Is the system stable? If $|w[i]| \leq \epsilon$, what can you say about $|x[i]|$ at all times i if you further assume that $u[i] = 0$ and the initial condition $x[0] = 0$? How big can $|x[i]|$ get?

(b) Suppose that we decide to choose a control law $u[i] = fx[i]$ to apply in feedback. **Given a specific λ, you want the system to behave like:**

$$x[i+1] = \lambda x[i] + w[i]? \tag{5}$$

To do so, how would you pick f?

NOTE: In this case, $w[i]$ can be thought of like another input to the system, except we can't control it.

$x[i+1] = 0.9x[i] + u[i] + w[i]$

$= 0.9x[i] + (fx[i]) + w[i]$

$x[i+1] = (0.9+f)x[i] + w[i]$

$x[i+1] = \lambda x[i] + w[i]$

λ determines BIBO stability

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use f to set λ

(c) **For the previous part, which f would you choose to minimize how big $|x[i]|$ can get?**

(d) **What if instead of a 0.9, we had a 3 in the original eq. (4). Would system stability change? Would our ability to control λ change?**

(e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \tag{6}$$

where we further assume that B is an invertible square matrix. Futher, suppose we decide to apply linear feedback control using a square matrix F so we choose $\vec{u}[i] = F\vec{x}[i]$.
Given a specific A_{CL} we want the system to behave like:

$$\vec{x}[i+1] = A_{\text{CL}}\vec{x}[i] + \vec{w}[i]? \tag{7}$$

How would you pick F given knowledge of A, B and the desired goal dynamics A_{CL}? Will this work for any desired A_{CL}?

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