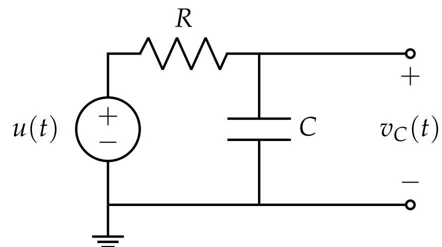
Discussion 8B

1. Stability Examples and Counterexamples

(a) Consider the circuit below with $R = 1\,\Omega$, $C = 0.5\,\text{F}$, and $u(t)$ is some function bounded between $-K$ and K for some constant $K \in \mathbb{R}$ (for example $K\cos(t)$). Furthermore assume that $v_C(0) = 0\,\text{V}$ (that the capacitor is initially discharged).



This circuit can be modeled by the differential equation

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = -2v_C(t) + 2u(t) \tag{1}$$

Show that the differential equation is always stable (that is, as long as the input $u(t)$ is bounded, $v_C(t)$ also stays bounded). Consider what this means in the physical circuit. *HINT: You may want to use the triangle inequality, i.e. $|a+b| \le |a| + |b|$, and the triangle inequality for integrals, i.e. $\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x$. When we use $|\cdot|$ notation here, we will take this to mean the magnitude, rather than the absolute value (since we can be dealing with complex numbers).*

Solution: We can apply the integral solution for a nonhomogeneous differential equation to demonstrate boundedness of the solution. The general solution to $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + bu(t)$ is $x(t) = x_0 e^{\lambda t} + \int_0^t e^{\lambda(t-\theta)} bu(\theta)\,\mathrm{d}\theta$. Here, we can say that:

$$v_C(t) = v_C(0)e^{-2t} + \int_0^t e^{-2(t-\theta)} 2u(\theta)\,\mathrm{d}\theta \tag{2}$$

$$= v_C(0)e^{-2t} + 2\int_0^t e^{-2(t-\theta)} u(\theta)\,\mathrm{d}\theta \tag{3}$$

We wish to show $|v_C(t)| \le M$ for all $t \ge 0$, where $M \in \mathbb{R}$ is some constant (this is another way to say that something is "bounded"). We can take the absolute value around eq. (3) as follows:

$$|v_C(t)| = \left| v_C(0)e^{-2t} + 2\int_0^t e^{-2(t-\theta)} u(\theta)\,\mathrm{d}\theta \right| \tag{4}$$

$$\le \left| v_C(0)e^{-2t} \right| + \left| 2\int_0^t e^{-2(t-\theta)} u(\theta)\,\mathrm{d}\theta \right| \tag{5}$$

$$\le \left| v_C(0)e^{-2t} \right| + 2\int_0^t \left| e^{-2(t-\theta)} u(\theta) \right| \mathrm{d}\theta \tag{6}$$

$$= |v_C(0)| e^{-2t} + 2\int_0^t e^{-2(t-\theta)} |u(\theta)|\,\mathrm{d}\theta \tag{7}$$

where we use the traditional triangle inequality to obtain eq. (5) and the integral triangle inequality to obtain eq. (6). We know $v_C(0) = 0$, so the first term is 0. Even if it is nonzero, we may assume that it is some finite constant. Furthermore, $0 \le e^{-2t} \le 1$ for $t \ge 0$ (it is a decaying exponential). Hence, the $|v_C(0)|e^{-2t}$ term is bounded. Next, we are allowed to assume that $|u(t)| \le K$ from the statement of the problem. This will let us obtain

$$|v_C(t)| \le 2\int_0^t e^{-2(t-\theta)} \underbrace{|u(\theta)|}_{\le K} \, d\theta \tag{8}$$

$$\le 2K\int_0^t e^{-2(t-\theta)} \, d\theta \tag{9}$$

$$= K\left(1 - e^{-2t}\right) \tag{10}$$

Because $e^{-2t} \ge 0$, $1 - e^{-2t} \le 1$. Hence, $|v_C(t)| \le K$ so $v_C(t)$ is bounded.

(b) **(PRACTICE)** Now, suppose that in the circuit of part 1.a we replaced the resistor with an inductor as in fig. 1.

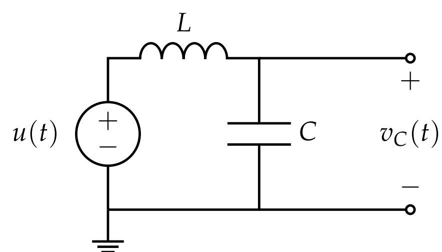


Figure 1: The original circuit with an inductor in place of the resistor.

Let $L = 1\,\text{mH}$. **Repeat part 1.a for the new circuit (with an inductor). Consider the following process to arrive at the result:**

i. **Derive the system of differential equations using KCL, KVL, and NVA. Show that the system is** $\frac{d}{dt}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$ **with the initial condition being** $\begin{bmatrix} v_C(0) \\ i_L(0) \end{bmatrix} = \vec{0}$**.**

ii. **Solve the matrix differential equation, using diagonalization if needed. Show that the diagonalized system has a solution**

$$\vec{y}(t) = \begin{bmatrix} \frac{1}{2LC} e^{j\frac{1}{\sqrt{LC}}t} \int_0^t e^{-j\frac{1}{\sqrt{LC}}\theta} u(\theta) \, d\theta \\ \frac{1}{2LC} e^{-j\frac{1}{\sqrt{LC}}t} \int_0^t e^{j\frac{1}{\sqrt{LC}}\theta} u(\theta) \, d\theta \end{bmatrix} \tag{11}$$

where $\vec{y}(t) = V^{-1}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$ **for change of basis matrix** V**.** You may use the fact that the eigenvalue, eigenvector pairs of $\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}$ are $\left(j\frac{1}{\sqrt{LC}}, \begin{bmatrix} -j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}\right)$ and $\left(-j\frac{1}{\sqrt{LC}}, \begin{bmatrix} j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}\right)$.

iii. **Apply a similar process from part 1.a to show that, if we have a bounded input** $u(t)$**, then the system can grow unboundedly.** When showing that a system is unstable, it suffices to choose a bounded $u(t)$ that makes the system unbounded. We can choose $u(t) =$

$2\cos\left(\frac{1}{\sqrt{LC}}\right) = e^{j\frac{1}{\sqrt{LC}}t} + e^{-j\frac{1}{\sqrt{LC}}t}$ [1]. *HINT: You may use the fact that $i_L(t) = y_1(t) + y_2(t)$.*

Hint: You might find it useful to revisit the process of generating the state-space equations for $v_C(t)$ and $i_L(t)$ as done in Note 4 for the LC Tank. The difference is that here, we have an input voltage.

Solution: 1.(b)i:

First, we begin forming the vector state-space equation, which involves relating $v_C(t)$ and $i_L(t)$ to their derivatives and the input voltage.

$$C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = i_C(t) = i_L(t) \tag{12}$$

$$\implies \frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = \frac{1}{C}i_L(t) \tag{13}$$

$$L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = v_L(t) = u(t) - v_C(t) \tag{14}$$

$$\implies \frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = \frac{1}{L}v_L(t) = -\frac{1}{L}v_C(t) + \frac{1}{L}u(t) \tag{15}$$

Combining this info, we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{\vec{b}}u(t) \tag{16}$$

1.(b)ii:

This is not a diagonal system, so we have to diagonalize it first. We start by solving for the eigenvalues and eigenvectors of A:

$$\lambda_1 = \mathrm{j}\frac{1}{\sqrt{LC}} \qquad \vec{v}_1 = \begin{bmatrix} -\mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \tag{17}$$

$$\lambda_2 = -\mathrm{j}\frac{1}{\sqrt{LC}} \qquad \vec{v}_1 = \begin{bmatrix} \mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \tag{18}$$

Note that these eigenvalues are purely imaginary. This will be helpful later. Our change of basis matrix is $V = \begin{bmatrix} -\mathrm{j}\sqrt{\frac{L}{C}} & \mathrm{j}\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix}$, so we can define our change of basis as $\vec{y}(t) = V^{-1}\vec{x}(t)$. Note that the new diagonal system will be

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{y}(t) = \begin{bmatrix} \mathrm{j}\frac{1}{\sqrt{LC}} & 0 \\ 0 & -\mathrm{j}\frac{1}{\sqrt{LC}} \end{bmatrix}\vec{y}(t) + V^{-1}\vec{b}u(t) \tag{19}$$

$$= \begin{bmatrix} \mathrm{j}\frac{1}{\sqrt{LC}} & 0 \\ 0 & -\mathrm{j}\frac{1}{\sqrt{LC}} \end{bmatrix}\vec{y}(t) + \left(\begin{bmatrix} -\mathrm{j}\sqrt{\frac{L}{C}} & \mathrm{j}\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}\right)u(t) \tag{20}$$

$$= \begin{bmatrix} \mathrm{j}\frac{1}{\sqrt{LC}} & 0 \\ 0 & -\mathrm{j}\frac{1}{\sqrt{LC}} \end{bmatrix}\vec{y}(t) + \begin{bmatrix} \frac{1}{2LC} \\ \frac{1}{2LC} \end{bmatrix}u(t) \tag{21}$$

[1]The natural frequency of this system is $\omega_n = \frac{1}{\sqrt{LC}}$. If we excite this system at a period equal to the natural frequency, we can make it grow unboundedly. This is similar to pushing a swing at the same rate it swings, which makes it swing farther.

so our system of equations is

$$\frac{\mathrm{d}}{\mathrm{d}t}y_1(t) = \mathrm{j}\frac{1}{\sqrt{LC}}y_1(t) + \frac{1}{2LC}u(t) \tag{22}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}y_2(t) = -\mathrm{j}\frac{1}{\sqrt{LC}}y_2(t) + \frac{1}{2LC}u(t) \tag{23}$$
$$\tag{24}$$

Recall that $\vec{x}(0) = \vec{0}$, so $\vec{y}(t) = \vec{0}$ (where $\vec{0}$ is a vector of all zeros). Solving this differential equation now, we get

$$y_1(t) = \underbrace{y_1(0)}_{0}\,\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t} + \int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}(t-\theta)}\left(\frac{1}{2LC}u(\theta)\right)\mathrm{d}\theta \tag{25}$$
$$y_2(t) = \underbrace{y_2(0)}_{0}\,\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t} + \int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}(t-\theta)}\left(\frac{1}{2LC}u(\theta)\right)\mathrm{d}\theta \tag{26}$$

Simplifying and stacking the solutions in vector form,

$$\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \vec{x}(t) = V\begin{bmatrix} \frac{1}{2LC}\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}\int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta}u(\theta)\,\mathrm{d}\theta \\ \frac{1}{2LC}\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}\int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta}u(\theta)\,\mathrm{d}\theta \end{bmatrix} \tag{27}$$

1.(b)iii:

We wish to show $\vec{x}(t)$ is unbounded, given some bounded input $u(t)$. When showing a vector is bounded, we can show that all of its individual, scalar entries are bounded. Alternatively, when showing a vector is unbounded, it is enough to show that one of its entries will be unbounded. Note that $i_L(t) = y_1(t) + y_2(t)$ (which we see by computing $\vec{x}(t) = V\vec{y}(t)$). We can show that this quantity is unbounded. Recall that

$$y_1(t) = \frac{\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta}u(\theta)\,\mathrm{d}\theta \tag{28}$$
$$y_2(t) = \frac{\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta}u(\theta)\,\mathrm{d}\theta \tag{29}$$
$$\implies i_L(t) = \frac{\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta}u(\theta)\,\mathrm{d}\theta + \frac{\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta}u(\theta)\,\mathrm{d}\theta \tag{30}$$

Now, we have to make some choice of a bounded input $u(t)$ so the entire term is unbounded as $t \to \infty$. We can choose $u(t) = \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t} + \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t} = 2\cos\left(\frac{1}{\sqrt{LC}}t\right)$ which is a bounded sinusoidal function. We can first compute $i_L(t)$ with this input:

$$i_L(t) = \frac{\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta}\left(\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta} + \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta}\right)\mathrm{d}\theta + \frac{\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta}\left(\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta} + \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta}\right)\mathrm{d}\theta \tag{31}$$

$$= \frac{\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t 1 + \mathrm{e}^{-\mathrm{j}\frac{2}{\sqrt{LC}}\theta}\,\mathrm{d}\theta + \frac{\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t 1 + \mathrm{e}^{\mathrm{j}\frac{2}{\sqrt{LC}}\theta}\,\mathrm{d}\theta \tag{32}$$
$$= \frac{\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\left(t + \frac{1 - \mathrm{e}^{-\mathrm{j}\frac{2}{\sqrt{LC}}t}}{\mathrm{j}\frac{2}{\sqrt{LC}}}\right) + \frac{\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\left(t + \frac{\mathrm{e}^{\mathrm{j}\frac{2}{\sqrt{LC}}t} - 1}{\mathrm{j}\frac{2}{\sqrt{LC}}}\right) \tag{33}$$

$$= \frac{t}{LC}\left(\frac{e^{j\frac{1}{\sqrt{LC}}t} + e^{-j\frac{1}{\sqrt{LC}}t}}{2}\right) + \frac{1}{\sqrt{LC}}\left(\frac{e^{j\frac{1}{\sqrt{LC}}t} - e^{-j\frac{1}{\sqrt{LC}}t}}{2j}\right) \tag{34}$$

$$= \frac{t}{LC}\cos\left(\frac{t}{\sqrt{LC}}\right) + \frac{1}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right) \tag{35}$$

Notice that the cos and sin terms are bounded, but the cos term is multiplied by a t, so as $t \to \infty$, $i_L(t) \to \infty$. Hence, the system is unstable. Generally, we say a system with eigenvalues having negative real part implies stability. Here, the real part of the eigenvalues is 0, so the system is unstable.

(c) Thus far, we have dealt with continuous systems so it also makes sense to consider discrete systems. Consider the discrete system

$$x[i+1] = 2x[i] + u[i] \tag{36}$$

with $x[0] = 0$.

Is the system stable or unstable? If unstable, find a bounded input sequence $u[i]$ that causes the system to "blow up".

Solution: Notice that, if we had the system

$$x[i+1] = 2x[i] \tag{37}$$

then we can write $x[i+1] = 2^i x[1]$. So, if we can somehow make $x[1]$ nonzero using a bounded input (e.g. equal to 1, for simplicity), then as $i \to \infty$, $x[i+1] \to \infty$. We know that $x[0] = 0$, and that $x[1] = 2x[0] + u[0] = u[0]$. Hence, we can set $u[0] = 1$ and then $x[1] = 1$. We have achieved what we wanted, i.e. to make $x[1]$ a nonzero value using the bounded input $u[0] = 1$. Now, for the other timesteps $i > 0$, we can set $u[i] = 0$ since that would leave us with the system in eq. (37). Written explicitly, our bounded input is

$$u[i] = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases} \tag{38}$$

2. Changing behavior through feedback

In this question, we discuss how *feedback control* can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[i+1] = 0.9x[i] + u[i] + w[i] \tag{39}$$

where $u[i]$ is the control input we get to apply based on the current state and $w[i]$ is the external disturbance, each at time i.

Is the system stable? If $|w[i]| \leq \epsilon$, what can you say about $|x[i]|$ at all times i if you further assume that $u[i] = 0$ and the initial condition $x[0] = 0$? How big can $|x[i]|$ get?

Solution: The system is stable, as $\lambda = 0.9 \implies |\lambda| < 1$. We can say that $|x[i]|$ is bounded at all time if the disturbance is bounded. Unrolling the system's recursion and extrapolating the general form,

$$x[0] = 0 \tag{40}$$
$$x[1] = w[0] \tag{41}$$
$$x[2] = 0.9w[0] + w[1] \tag{42}$$
$$x[3] = 0.9^2 w[0] + 0.9w[1] + w[2] \tag{43}$$
$$\vdots \tag{44}$$
$$x[i] = \sum_{k=0}^{i-1} 0.9^k w[i-k-1]. \tag{45}$$

We can check that this form works by plugging it into our recursion:

$$x[i+1] = 0.9x[i] + w[i] = 0.9\left(\sum_{k=0}^{i-1} 0.9^k w[i-k-1]\right) + w[i] \tag{46}$$
$$= \sum_{k=0}^{i-1} 0.9^{k+1} w[i-k-1] + w[i] = \sum_{k=0}^{i} 0.9^k w[i-k] \tag{47}$$

which is exactly what our formula predicts. So,

$$|x[i]| = \left|\sum_{k=0}^{i-1} 0.9^k w[i-k-1]\right| \leq \sum_{k=0}^{i-1} \left|0.9^k w[i-k-1]\right| = \sum_{k=0}^{i-1} 0.9^k \epsilon. \tag{48}$$

In the limit as $i \to \infty$, by the geometric series formula,

$$|x[i]| \leq \frac{\epsilon}{1-0.9} = 10\epsilon \tag{49}$$

(b) Suppose that we decide to choose a control law $u[i] = fx[i]$ to apply in feedback. **Given a specific λ, you want the system to behave like:**

$$x[i+1] = \lambda x[i] + w[i]? \tag{50}$$

To do so, how would you pick f?

NOTE: In this case, $w[i]$ can be thought of like another input to the system, except we can't control it.

Solution: We can control the system to have any value of λ, as long as we're not limited on the values of f.

$$x[i+1] = 0.9x[i] + fx[i] + w[i] = \lambda x[i] + w[i]. \tag{51}$$

Fitting terms, $f = \lambda - 0.9$. Note we can get a $\lambda > 1$ if we so desire; there is nothing stopping us from putting arbitrarily big/small λ by the choice of f.

(c) **For the previous part, which f would you choose to minimize how big $|x[i]|$ can get?**

Solution: From eq. (50), in order to have the minimum bound on $|x[i]|$, $\lambda = 0$. To get this λ, $f = -0.9$. In the limit as $i \to \infty$ in this case,

$$|x[i]| \leq \frac{\epsilon}{1-0} = \epsilon \tag{52}$$

The minimum bound on $|x(i)| = \epsilon$ is the same bound as on the disturbance.

(d) **What if instead of a 0.9, we had a 3 in the original eq. (39). Would system stability change? Would our ability to control λ change?**

Solution: If our system were now,

$$x[i+1] = 3x[i] + u[i] + w[i], \tag{53}$$

the system would no longer be stable. However, we can still choose any λ using closed loop feedback. In this case, $f = \lambda - 3$.

(e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \tag{54}$$

where we further assume that B is an invertible square matrix. Futher, suppose we decide to apply linear feedback control using a square matrix F so we choose $\vec{u}[i] = F\vec{x}[i]$.

Given a specific A_{CL} we want the system to behave like:

$$\vec{x}[i+1] = A_{\text{CL}}\vec{x}[i] + \vec{w}[i]? \tag{55}$$

How would you pick F given knowledge of A, B and the desired goal dynamics A_{CL}? Will this work for any desired A_{CL}?

Solution: Since in this case our input is the same rank as our output, we can arbitrarily choose the matrix A_{CL}. As long as B is invertible (as given), we can define:

$$\begin{aligned}
\vec{x}[i+1] &= A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] && (56)\\
&= A\vec{x}[i] + BF\vec{x}[i] + \vec{w}[i] && (57)\\
&= (A + BF)\vec{x}[i] + \vec{w}[i] && (58)\\
&= A_{\text{CL}}\vec{x}[i] + \vec{w}[i] && (59)
\end{aligned}$$

Therefore, matching terms,

$$A + BF = A_{\text{CL}} \implies F = B^{-1}(A_{\text{CL}} - A). \tag{60}$$

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