Discussion 9A

1. **Eigenvalue Placement in Discrete Time**

   Recall that, for a discrete linear system to be stable, we require that all of the eigenvalues of A in $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$ must have magnitude less than 1.

   Consider the following linear discrete time system

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + \vec{w}[i] \tag{1}$$

   (a) **Is the system given in eq. (1) stable?**

$$|\lambda_i| < 1 \quad \forall \lambda_i$$

$$\det\left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix}\right)$$

$\det(A-\lambda I)=0$

$$= \lambda(1+\lambda) - 2$$
$$= \lambda^2+\lambda-2$$
$$= (\lambda+2)(\lambda-1) = 0$$
$$\Rightarrow \lambda = -2, 1$$

   (b) We can attempt to stabilize the system by implementing closed loop feedback. That is, we choose our input $u[i]$ so that the system is stable. **If we were to use state feedback as in eq. (2), what is an equivalent representation for this system? Write your answer as $\vec{x}[i+1] = A_{\mathrm{CL}}\vec{x}[i]$ for some matrix A_{CL}.**

$$u[i] = \overbrace{\begin{bmatrix} f_1 & f_2 \end{bmatrix}}^{f^T} \vec{x}[i] \tag{2}$$

   HINT: If you're having trouble parsing the expression for $u[i]$, note that $\begin{bmatrix} f_1 & f_2 \end{bmatrix}$ is a row vector, while $\vec{x}[i]$ is column vector. What happens when we multiply a row vector with a column vector like this?)

$$x[i+1] = Ax[i] + Bu[i]$$
$$= Ax[i] + Bf^T x[i]$$
$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x[i] + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} f_1 & f_2 \end{bmatrix}} x[i]$$
$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x[i] + \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix} x[i]$$
$$= (A + Bf^T)x[i]$$

$$\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}$$
$(2\times1)\ (1\times2) \rightarrow (2\times2)$

$$x[i+1] = \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{CL}} x[i]$$

(c) **Find the appropriate state feedback constants, f_1, f_2, that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.**

$$x[i+1] = \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{CL}} x[i]$$

$$\det(A_{CL} - \lambda I) = \det\left(\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} f_1-\lambda & 1+f_2 \\ 2 & -1-\lambda \end{bmatrix}\right)$$

$$= (f_1-\lambda)(-1-\lambda) - 2(1+f_2)$$
$$= (\lambda - f_1)(\lambda+1) - 2(1+f_2)$$
$$= \lambda^2 + (1-f_1)\lambda - f_1 - 2(1+f_2)$$
$$= \lambda^2 + (1-f_1)\lambda - f_1 - 2 - 2f_2 \qquad \lambda^2 + \alpha\lambda + \beta$$

$$(\lambda + 1/2)(\lambda - 1/2)$$
$$= \lambda^2 + 0\cdot\lambda - 1/4$$

$$1 - f_1 = 0$$
$$-f_1 - 2 - 2f_2 = -1/4$$

$$\left|\tfrac{1}{3} e^{j\pi/3}\right| = \left|\tfrac{1}{3}\right| \left|e^{j\pi/3}\right| = 1/3$$

(d) **Is the system now stable in closed-loop, using the control feedback coefficients f_1, f_2 that we derived above?**

(e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i]$ in eq. (1), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]$ as the way that the discrete-time control acted on the system. In other words, the system is as given in eq. (3).

$$\vec{x}[i+1] = \overset{A}{\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}} \vec{x}[i] + \overset{B}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} u[i] \qquad \overset{AB}{(2\times2)(2\times1)} \to (2\times1) \tag{3}$$

Determine whether the system is controllable or not.

a system is controllable if

$$C = \begin{bmatrix} | & | \\ AB & B \\ | & | \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

C has full rank

full rank $\Leftrightarrow$ lin. ind. columns

(f) Let's say we still try and apply closed loop feedback to our system. Let's use $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]$ to try and control the system. **Show that the resulting closed-loop state space matrix is**

$$A_{\text{CL}} = \begin{bmatrix} f_1 & f_2+1 \\ f_1+2 & f_2-1 \end{bmatrix} \tag{4}$$

Is it possible to stabilize this system?

$$\det(A_{CL} - \lambda I) = \det\left(\begin{bmatrix} f_1 & f_2+1 \\ f_1+2 & f_2-1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} f_1-\lambda & f_2+1 \\ f_1+2 & f_2-1-\lambda \end{bmatrix}\right)$$

$$= (f_1-\lambda)(f_2-1-\lambda) - (f_1+2)(f_2+1)$$

$$= [f_1 f_2 - f_1 - f_1\lambda - f_2\lambda + \lambda + \lambda^2] - [f_1 f_2 + f_1 + 2f_2 + 2]$$

$$= \lambda^2 + (1-f_1-f_2)\lambda + f_1 f_2 - f_1 - f_1 f_2 - f_1 - 2f_2 - 2$$

$$= \lambda^2 + (1-f_1-f_2)\lambda - 2f_1 - 2f_2 - 2$$

$$= (\lambda+2)(\lambda - (1+f_1+f_2))$$

Δ B

no longer place all λ

2. **Uncontrollability**

Recall that, for a n-dimensional, discrete-time linear system to be controllable, we require that the controllability matrix $\mathcal{C} = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \dots & AB & B \end{bmatrix}$ to be rank n.

Consider the following discrete-time system with the given initial state:

$$\vec{x}[i+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i] \tag{5}$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

(a) **Is the system controllable?**

$A \cdot A \cdot B$

$(3\times3)(3\times3)(3\times1) \rightarrow (3\times1)$

$$C = \begin{bmatrix} | & | & | \\ A^2B & AB & B \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

if C is full rank

where can C take us?

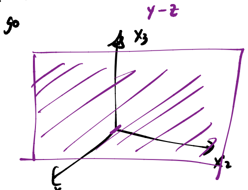
answer by introducing span

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \end{bmatrix} \rightarrow \text{where we can go}$$

$\text{col}(C) \in \mathbb{R}^3$

Reachability

y-z



(b) **Show that we can write the** i**th state as**

$$\vec{x}[i] = \begin{bmatrix} 2^i \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \tag{7}$$

Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ **for some** ℓ**? If so, for what input sequence** $u[i]$ **up to** $i = \ell - 1$**?**

$$\vec{x}[i] = \begin{bmatrix} x_1[i] \\ x_2[i] \\ x_3[i] \end{bmatrix}$$

(c) **Is it possible to reach** $\vec{x}[\ell] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ **for some** ℓ**? For what input sequence** $u[i]$ **for** $i = 0$ **to** $i = \ell - 1$**?**

HINT: Use the result for $\vec{x}[i]$ *from the previous part.*

(d) **Find the set of all $\vec{x}[2]$, given that you are free to choose any $u[0]$ and $u[1]$.**

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