## Agenda:

- Midterm Redo due 3/22
  - → grades released shortly after
- HW 9 - due 3/24
- Attend Lecture - EC opportunity :)
- Controllability / Reachability
- Worksheet

Last Time: We can control system via feedback control to place our eigenvalues at certain values

→ Can we always apply this technique? (aka. is our system controllable?)

## Definitions:

Reachability: possible to provide inputs that push model state to some target state given some initial state

Controllability: model can reach any given target state from any initial state

Controllability is stricter than reachability

Recall for DT:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$$

$$\Rightarrow \vec{x}[i] = A^i\vec{x}[0] + \sum_{k=0}^{i-1} A^{i-1-k} B\vec{u}[k] \quad \text{(state trajectory)}$$

can write as a dot product

$$\Rightarrow \vec{x}[i] = A^i\vec{x}[0] + \underbrace{\begin{bmatrix} A^{i-1}B & A^{i-2}B & \dots & AB & B \end{bmatrix}}_{\text{Define this as Controllability Matrix}} \begin{bmatrix} \vec{u}[0] \\ \vec{u}[1] \\ \vdots \\ \vec{u}[i-2] \\ \vec{u}[i-1] \end{bmatrix}$$

Def: Controllability Matrix @ timestep $i$ ☆☆

$$C_i = \begin{bmatrix} A^{i-1}B & A^{i-2}B & \dots & AB & B \end{bmatrix}$$

$$\Rightarrow \vec{x}[i] = A^i\vec{x}[0] + C_i \underbrace{\begin{bmatrix} \vec{u}[0] \\ \vec{u}[1] \\ \vdots \\ \vec{u}[i-2] \\ \vec{u}[i-1] \end{bmatrix}}_{\text{we choose these values}}$$

⇒ we can "reach" anything in the span of $C_i$

## Reachability:

- Given fixed initial state $\vec{x}_0 \in \mathbb{R}^n$
- Given fixed target state $\vec{x}^* \in \mathbb{R}^n$

→ $\vec{x}^*$ is reachable in $i^*$ timesteps from $\vec{x}_0$

$\Updownarrow$

$\vec{x}^* - A^i\vec{x}_0 \in Col(C_{i^*})$ — span {columns of $C_i$}

(1) Solve using Gaussian Elimination
→ may exist multiple solutions

(2) If no solutions exist ⇒ not reachable

## Controllability:

- Given any initial state $\vec{x}_0 \in \mathbb{R}^n$
- Given any target state $\vec{x}_* \in \mathbb{R}^n$

Instead of:

$\underbrace{\vec{x}^* - A^i\vec{x}_0}_{\text{this can be any vector in } \mathbb{R}^n} \in Col(C_i)$ (reachability)

⇒ need span of columns of $C_i$ to be $\mathbb{R}^n$

$C_n = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \dots AB & A \end{bmatrix}$ where $\vec{x}[i] \in \mathbb{R}^n$

☆☆ (1) Controllable in $i$ timesteps ⇒ $Col(C_i) = \mathbb{R}^n$

☆☆ (2) Controllable (no time constraint) ⇒ $Col(C_n) = \mathbb{R}^n$