

Discussion 9A

1. Eigenvalue Placement in Discrete Time

Recall that, for a discrete linear system to be stable, we require that all of the eigenvalues of A in $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$ must have magnitude less than 1.

Consider the following linear discrete time system

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + \vec{w}[i] \quad (1)$$

(a) **Is the system given in eq. (1) stable?**

Solution: For notation's sake, let's write the system in the familiar form

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i] + \vec{w}[i] \quad (2)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (3)$$

We have to calculate the eigenvalues of matrix A . Doing so, we find:

$$\det(A - \lambda I) = 0 \implies \lambda_1 = 1, \lambda_2 = -2 \quad (4)$$

Since there exists a λ such that $|\lambda| \geq 1$ (in fact, both λ_1 and λ_2 satisfy this inequality), the system is unstable.

(b) We can attempt to stabilize the system by implementing closed loop feedback. That is, we choose our input $u[i]$ so that the system is stable. **If we were to use state feedback as in eq. (5), what is an equivalent representation for this system? Write your answer as $\vec{x}[i+1] = A_{CL}\vec{x}[i]$ for some matrix A_{CL} .**

$$u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \quad (5)$$

HINT: If you're having trouble parsing the expression for $u[i]$, note that $\begin{bmatrix} f_1 & f_2 \end{bmatrix}$ is a row vector, while $\vec{x}[i]$ is column vector. What happens when we multiply a row vector with a column vector like this?

Solution: The closed loop system using state feedback has the form

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] \quad (6)$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \right) \quad (7)$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] \quad (8)$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix} \right) \vec{x}[i] \quad (9)$$

$$= \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{CL}} \vec{x}[i]. \quad (10)$$

- (c) Find the appropriate state feedback constants, f_1, f_2 , that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

Solution: From the previous part we have computed the closed loop system as

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{CL}} \vec{x}[i] \quad (11)$$

Thus, finding the eigenvalues of the above system we have

$$0 = \det(A - \lambda I) \quad (12)$$

$$= \det \left(\begin{bmatrix} f_1 - \lambda & 1+f_2 \\ 2 & -1-\lambda \end{bmatrix} \right) \quad (13)$$

$$= \lambda^2 + (1-f_1)\lambda + (-f_1 - 2f_2 - 2) \quad (14)$$

We want to place the eigenvalues at $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = \frac{1}{2}$. This means that we should choose the constants f_1 and f_2 so that the characteristic equation is

$$0 = \left(\lambda - \frac{1}{2} \right) \left(\lambda + \frac{1}{2} \right) = \lambda^2 - \frac{1}{4} = \lambda^2 + 0\lambda - \frac{1}{4} \quad (15)$$

Thus, we can match the coefficients of λ in the polynomial above, which indicates we should choose f_1 and f_2 satisfying the following system of equations:

$$0 = 1 - f_1 \quad (16)$$

$$-\frac{1}{4} = -f_1 - 2f_2 - 2 \quad (17)$$

We can solve this two variable, two equation system and find that $f_1 = 1, f_2 = -\frac{11}{8}$.

Alternatively, we know what the eigenvalues are; we can plug in each λ into characteristic polynomial, and doing so will yield the same system of equations in f_1, f_2 .

- (d) Is the system now stable in closed-loop, using the control feedback coefficients f_1, f_2 that we derived above?

Solution: Yes, the closed loop system has eigenvalues $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$, which means that A_{CL} satisfies our condition that all of its eigenvalues have magnitude less than 1.

- (e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i]$ in eq. (1), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]$ as the way that the discrete-time control acted on the system. In other words, the system is as given in eq. (18).

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i] \quad (18)$$

Determine whether the system is controllable or not.

Solution:

$$C = [A\vec{b} \quad \vec{b}] \quad (19)$$

$$A\vec{b} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (20)$$

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (21)$$

As we can see, the controllability matrix is not full rank and therefore the system is not controllable.

- (f) Let's say we still try and apply closed loop feedback to our system. Let's use $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]$ to try and control the system. **Show that the resulting closed-loop state space matrix is**

$$A_{CL} = \begin{bmatrix} f_1 & f_2 + 1 \\ f_1 + 2 & f_2 - 1 \end{bmatrix} \quad (22)$$

Is it possible to stabilize this system?

Solution:

$$\vec{x}[i+1] = \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] \quad (23)$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] \quad (24)$$

$$= \underbrace{\begin{bmatrix} f_1 & f_2 + 1 \\ f_1 + 2 & f_2 - 1 \end{bmatrix}}_{A_{CL}} \vec{x}[i] \quad (25)$$

Finding the eigenvalues λ :

$$0 = \det \left(\begin{bmatrix} f_1 - \lambda & f_2 + 1 \\ f_1 + 2 & f_2 - 1 - \lambda \end{bmatrix} \right) \quad (26)$$

$$= (f_1 - \lambda)(f_2 - 1 - \lambda) - (f_1 + 2)(f_2 + 1) \quad (27)$$

$$= f_1(f_2 - 1) - f_1\lambda - \lambda(f_2 - 1) + \lambda^2 - (f_1f_2 + f_1 + 2f_2 + 2) \quad (28)$$

$$= f_1f_2 - f_1 - f_1\lambda - \lambda f_2 + \lambda + \lambda^2 - f_1f_2 - f_1 - 2f_2 - 2 \quad (29)$$

$$= \lambda^2 + (1 - f_1 - f_2)\lambda - 2(1 + f_1 + f_2) \quad (30)$$

$$= (\lambda + 2)(\lambda - (1 + f_1 + f_2)) \quad (31)$$

We can see that the eigenvalue at $\lambda = -2$ cannot be moved, so we cannot arbitrarily change our eigenvalues with this control input. Since there will always be an eigenvalue with $|\lambda| \geq 1$, then we cannot stabilize the system.

2. Uncontrollability

Recall that, for a n -dimensional, discrete-time linear system to be controllable, we require that the controllability matrix $C = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \dots & AB & B \end{bmatrix}$ to be rank n .

Consider the following discrete-time system with the given initial state:

$$\vec{x}[i+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i] \quad (32)$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (33)$$

(a) **Is the system controllable?**

Solution:

$$\mathcal{C} = \begin{bmatrix} A^2B & AB & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \quad (34)$$

Since the controllability matrix \mathcal{C} only has rank 2, the system is not controllable. We would need it to be rank 3 here to span the full space \mathbb{R}^3 .

(b) **Show that we can write the i th state as**

$$\vec{x}[i] = \begin{bmatrix} 2^i \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \quad (35)$$

Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some ℓ ? If so, for what input sequence $u[i]$ up to $i = \ell - 1$?

Solution: We can write:

$$\vec{x}[i] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i-1] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i-1] \quad (36)$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[i-1] \\ x_2[i-1] \\ x_3[i-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i-1] \quad (37)$$

$$= \begin{bmatrix} 2x_1[i-1] \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \quad (38)$$

$$= \begin{bmatrix} 2^i x_1[0] \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \quad (39)$$

$$= \begin{bmatrix} 2^i \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \quad (40)$$

Note that in this expression for $\vec{x}[i]$, $x_1[i] = 2^i$ is decoupled from all other states and inputs. From this expression we also see that there is no choice of inputs us to get to $x_1[\ell] = -2$. Therefore,

we will never be able to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for any ℓ .

- (c) **Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some ℓ ? For what input sequence $u[i]$ for $i = 0$ to $i = \ell - 1$?**

HINT: Use the result for $\vec{x}[i]$ from the previous part.

Solution: We need $\ell = 1$ since $x_1[i] = 2^i x_1[0] = 2^i$. Hence,

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} \quad (41)$$

We realize that the first two entries of $\vec{x}[1]$ are exactly what we want. Thus, we have to choose $u[0]$ so the third entry is -2 . If we choose $u[0] = -1$, then we have reached our desired state.

Thus we see that a system being uncontrollable does not mean we are unable to reach anything at all, but just that the range that can be reached is limited.

- (d) **Find the set of all $\vec{x}[2]$, given that you are free to choose any $u[0]$ and $u[1]$.**

Solution:

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} \quad (42)$$

$$\vec{x}[2] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[1] \quad (43)$$

$$= \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix} \quad (44)$$

Since we can set $u[0]$ and $u[1]$ arbitrarily, we can reach any state of the form $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \text{span} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

after two timesteps. This means that we can reach any value for $\vec{x}[2]$; contrast this with how the first component of the state vector is fixed at 4 after two timesteps, and cannot be changed by the inputs.

Alternative Solution:

Notice that we can write

$$\vec{x}[2] = A^2 \vec{x}[0] + ABu[0] + Bu[1] = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix} \quad (45)$$

Hence, $\vec{x}[2]$ will be $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$ plus whatever is in the column space of

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \quad (46)$$

This gives the same answer as before, i.e. that

$$\vec{x}[2] = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \vec{p} \quad (47)$$

where $\vec{p} \in \text{span} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

Any desired $x_2[i]$ and $x_3[i]$ that we can possibly reach can be obtained in only two or fewer timesteps. Hence, every reachable state can be written as

$$\vec{x}[i] = \begin{bmatrix} 2^i \\ 0 \\ 0 \end{bmatrix} + \vec{p} \quad (48)$$

with \vec{p} defined as above. This will also tell us why the desired goal in part 2.b is unreachable.

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