Agenda:

- Midterm Redo due tonight!
- HW9 - due 3/24
- Mid-semester Survey due tonight
- Exam scores released!
  - Regrades + solutions tonight or tmrw
- Orthonormality
- Gram Schmidt/Orthonormalization
- Worksheet
- Enjoy your spring break!

Goals:

(1) Understand benefits of orthonormality
(2) Learn how to find orthonormal basis (Gram-Schmidt)

$\boxed{\text{Span Review}}$

if $\vec{v} \in \text{Span}\{\vec{s}_1, \vec{s}_2\}$ then

$\vec{v} = \alpha\vec{s}_1 + \beta\vec{s}_2$ for some $\alpha, \beta \in \mathbb{R}$

$\Rightarrow \vec{v}$ can be written as a linear combination of α, β.

$\boxed{\text{Orthonormality}}$ $\begin{cases} 1) \text{ orthogonal} \\ 2) \text{ normal} \end{cases}$

1) Given $\vec{x}, \vec{y} \in \mathbb{R}^n$:

- $\vec{x}$ and $\vec{y}$ are orthogonal $\Leftrightarrow \langle \vec{x}, \vec{y} \rangle = \vec{y}^T\vec{x} = 0$

set of vectors $S = \{\vec{s}_1 \dots \vec{s}_m\}$ if

$\langle \vec{s}_i, \vec{s}_j \rangle = 0$ for all $\vec{s}_i \neq \vec{s}_j$

2) Normalized vectors: unit norm, $\|\vec{x}\| = 1$

Orthonormal sets of vectors (S)

(1) All vectors unit norm
(2) All vectors orthogonal to each other

Mathematically:

$$\langle \vec{x}, \vec{y} \rangle = \begin{cases} 1 & \text{if } \vec{x} = \vec{y} \\ 0 & \text{if } \vec{x} \neq \vec{y} \end{cases} \quad \text{for all } \vec{x}, \vec{y} \in S$$

(1) Square $Q \in \mathbb{R}^{n \times n}$ (column & rows are orthonormal sets)

- "orthonormal matrix"
- $Q^TQ = I_n$, $QQ^T = I_n$, $Q^T = Q^{-1}$

(2) Tall $Q \in \mathbb{R}^{m \times n}$ $(m \geq n)$ (columns are orthonormal set)

- "orthonormal columns"
- $Q^TQ = I_n$

(3) Wide $Q \in \mathbb{R}^{m \times n}$ $(m \leq n)$ (rows are orthonormal set)

- "orthonormal rows"
- $QQ^T = I_m$

$\boxed{\text{Gram-Schmidt Orthonormalization}}$ QI

- process for converting set of vectors/matrix into orthonormal set of vectors w/ same span (matrix w/ orthonormal columns w/ same columnspace)

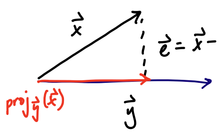
$S = \{\vec{s}_1, \dots, \vec{s}_n\} \Rightarrow Q = \{\vec{q}_1, \dots, \vec{q}_n\}$ where

① Q is an orthonormal set of vectors
② $\text{span}\{\vec{s}_1, \dots, \vec{s}_n\} = \text{span}\{\vec{q}_1, \dots, \vec{q}_n\}$

Main Idea:

(1) Normalize first vector & add to Q
(2) For each subsequent vector:
   (a) Project vector onto all vectors in Q
   (b) Subtract projections from vector
   (c) Normalize new vector and add to Q
(3) Repeat step 2 until desired subspace is reached

Projections Review: Goal: Find component of $\vec{x}$ in direction of $\vec{y}$.



$\vec{e} = \vec{x} - \text{proj}_{\vec{y}}(\vec{x}) = \vec{x} - \underbrace{c\vec{y}}_{}$

some scalar multiple of $\vec{y}$ (span of $\vec{y}$)

We know $\vec{e} \perp \vec{y} \Rightarrow \langle \vec{e}, \vec{y} \rangle = 0$

$\langle \vec{x} - c\vec{y}, \vec{y} \rangle = 0$

$\langle \vec{x}, \vec{y} \rangle - c\langle \vec{y}, \vec{y} \rangle = 0$

$c = \dfrac{\langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle}$

Therefore,

$$\text{proj}_{\vec{y}}(\vec{x}) = c\vec{y} = \frac{\langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle}\vec{y} = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}\vec{y} \quad ☆☆$$

$\rightarrow$ if $\vec{y}$ is normalized $\|\vec{y}\|^2 = 1 \Rightarrow \text{proj}_{\vec{y}}(\vec{x}) = \langle \vec{x}, \vec{y} \rangle \vec{y}$