Agenda: tinyurl.com/sp23-oliver-resources

- Welcome back!
- Regrade Requests due Friday 4/7
- Status Check Released
- Recap:
  - Orthonormality
  - Gram Schmidt
- Today: More Orthonormality

☆ Taken directly from Dis 9B notes

$\boxed{\text{Orthonormality}}$ $\begin{cases} 1) \text{ orthogonal} \\ 2) \text{ normal} \end{cases}$

1) Given $\vec{x}, \vec{y} \in \mathbb{R}^n$:

- $\vec{x}$ and $\vec{y}$ are orthogonal $\Leftrightarrow \langle \vec{x}, \vec{y} \rangle = \vec{y}^T \vec{x} = 0$

Set of vectors $S = \{\vec{s}_1 \dots \vec{s}_m\}$ if

$\langle \vec{s}_i, \vec{s}_j \rangle = 0$ for all $\vec{s}_i \neq \vec{s}_j$

2) Normalized vectors: unit norm, $\|\vec{x}\| = 1$

Orthonormal sets of Vectors (S)

(1) All vectors unit norm

(2) All vectors orthogonal to each other

Mathematically:

$$\langle \vec{x}, \vec{y} \rangle = \begin{cases} 1 & \text{if } \vec{x} = \vec{y} \\ 0 & \text{if } \vec{x} \neq \vec{y} \end{cases} \quad \text{for all } \vec{x}, \vec{y} \in S$$

(1) Square $Q \in \mathbb{R}^{n \times n}$ (column & rows are orthonormal sets)

- "orthonormal matrix"
- $Q^T Q = I_n$, $QQ^T = I_n$, $Q^T = Q^{-1}$

(2) Tall $Q \in \mathbb{R}^{m \times n}$ $(m \geq n)$ (columns are orthonormal set)

- "orthonormal columns"
- $Q^T Q = I_n$

(3) Wide $Q \in \mathbb{R}^{m \times n}$ $(m \leq n)$ (rows are orthonormal set)

- "orthonormal rows"
- $QQ^T = I_m$

$\boxed{\text{Gram Schmidt Review}}$ (Dis 9B)

Goal: Given set of vectors $S = \{\vec{s}_1, \vec{s}_2, \dots \vec{s}_m\}$, find an orthonormal set of vectors $Q = \{\vec{q}_1, \vec{q}_2, \dots \vec{q}_m\}$ s.t. $\text{Span}\{Q\} = \text{Span}\{S\}$

① $\vec{q}_1 = \frac{\vec{s}_1}{\|\vec{s}_1\|}$ (just normalize)

② $\vec{q}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|}$ where $\vec{z}_2 = \vec{s}_2 - \langle \vec{s}_2, \vec{q}_1 \rangle \vec{q}_1$

③ $\vec{q}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|}$ where $\vec{z}_3 = \vec{s}_3 - \langle \vec{s}_3, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{s}_3, \vec{q}_2 \rangle \vec{q}_2$

④ $\vec{q}_4 = \frac{\vec{z}_4}{\|\vec{z}_4\|}$ where $\vec{z}_4 = \vec{s}_4 - \underbrace{\langle \vec{s}_4, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{s}_4, \vec{q}_2 \rangle \vec{q}_2 - \langle \vec{s}_4, \vec{q}_3 \rangle \vec{q}_3}_{\text{subtract projections of } \vec{s}_i \text{ onto all previous vectors in set}}$

Recall: $\vec{z}_i$ represents component of $\vec{s}_i$ that is orthogonal to all previous vectors in set

⋮

ⓘ $\vec{q}_i = \frac{\vec{z}_i}{\|\vec{z}_i\|}$ where $\vec{z}_i = \vec{s}_i - \sum_{j=1}^{i-1} \langle \vec{s}_i, \vec{q}_j \rangle \vec{q}_j$

"Basis Extension" via Gram-Schmidt (Algorithm 24 in Note 11)

Problem: starting w/ some set of vectors S which span a subspace of $\mathbb{R}^n$, we want to find an orthonormal basis for $\mathbb{R}^n$ that contains the basis vectors of the original S

Idea: append some basis of $\mathbb{R}^n$ to S & run GS on all vectors

⇒ Use standard basis vectors! ☆☆

Ex: Suppose you want to extend an orthonormal basis of $\mathbb{R}^3$ from $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

→ First notice than $\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\} \neq \mathbb{R}^3$

→ To extend basis, add standard basis of $\mathbb{R}^3$ & run GS

$$GS\left(\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{appended standard basis vectors}} \right\} \right) \Rightarrow \text{basis for } \mathbb{R}^3$$