EECS 16B Designing Information Systems and Devices II UC Berkeley Spring 2023

# Discussion 10B

**1. $2 \times 2$ Upper Triangularization Example**

Previously in this course, we have seen the value of changing our coordinates to be eigenbasis-aligned, because we can then view the system as a set of parallel scalar systems. Diagonalization causes these scalar equations to be fully uncoupled such that they can be solved separately. But even when we cannot diagonalize, we can *upper-triangularize* such that we can still solve the equations one at a time, from the "bottom up".

Recall that Schur Decomposition is a method by which we can take some $M$ matrix and decompose it into $U^\top T U$ where $U$ is an orthonormal matrix and $T$ is an upper triangular matrix. This is the Real Schur Decomposition algorithm from Note 15 for reference.

**Algorithm 1** Real Schur Decomposition

**Require:** A square matrix $A \in \mathbb{R}^{n \times n}$ with real eigenvalues.

**Ensure:** An orthonormal matrix $U \in \mathbb{R}^{n \times n}$ and an upper-triangular matrix $T \in \mathbb{R}^{n \times n}$ such that $A = UTU^\top$.

**function** $\textsc{RealSchurDecomposition}(A)$

  **if** $A$ is $1 \times 1$ **then**

    **return** $\begin{bmatrix} 1 \end{bmatrix}, A$

  **end if**

  $(\vec{q}_1, \lambda_1) := \textsc{FindEigenvectorEigenvalue}(A)$

  $Q := \textsc{ExtendBasis}(\{\vec{q}_1\}, \mathbb{R}^n)$ ▷ Extend $\{\vec{q}_1\}$ to a basis of $\mathbb{R}^n$ using Gram-Schmidt; see Note 13

  Unpack $Q := \begin{bmatrix} \vec{q}_1 & \widetilde{Q} \end{bmatrix}$

  Compute and unpack $Q^\top A Q = \begin{bmatrix} \lambda_1 & \vec{\tilde{a}}_{12}^\top \\ \vec{0}_{n-1} & \widetilde{A}_{22} \end{bmatrix}$

  $(P, \widetilde{T}) := \textsc{RealSchurDecomposition}(\widetilde{A}_{22})$

  $U := \begin{bmatrix} \vec{q}_1 & \widetilde{Q} P \end{bmatrix}$

  $T := \begin{bmatrix} \lambda_1 & \vec{\tilde{a}}_{12}^\top P \\ \vec{0}_{n-1} & \widetilde{T} \end{bmatrix}$

  **return** $(U, T)$

**end function**

In this problem, we are going to be working with the following $2 \times 2$ matrix:

$$A = \begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix} \quad \text{not diagonalizable} \tag{1}$$

(a) Remember that diagonalization is another tool we have learned that can be used to decompose a matrix. However, there is the restriction that our transformation $V$ (which was chosen to be a matrix of the eigenvectors of the $A$ matrix) must be invertible. That means that we needed $n$ linearly independent eigenvectors for a matrix $A \in \mathbb{R}^{n \times n}$. **For the given matrix $A$, calculate its eigenvalues and eigenvectors and determine whether or not we can diagonalize the matrix.**

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -8-\lambda & -5 \\ 5 & 2-\lambda \end{bmatrix}\right) = \underbrace{(-8-\lambda)(2-\lambda)}_{} - (-25) = 0$$

$$-16 - 2\lambda + 8\lambda + \lambda^2 + 25 = 0$$

$$\lambda^2 + 6\lambda + 9 = 0$$

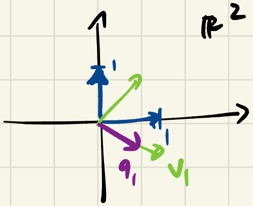
$$(\lambda + 3)^2 = 0$$

$$\lambda = -3$$

$$V = \begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix}$$

$$\text{null}\left(\begin{bmatrix} -5 & -5 \\ 5 & 5 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(b) Hopefully, in the previous part you observed that this matrix has repeated eigenvalues and as a result did not have linearly independent eigenvectors. Instead let's try and upper triangularize the system. Recall that the first step of Schur Decomposition is to calculate an eigenvalue, eigenvector pair. We have already done that in 1.a, so we can directly use our calculations.
**Using Gram-Schmidt, extend an orthogonal basis for $\mathbb{R}^2$ from our eigenvector $\vec{v}_1$. In other words, find an orthonormal set of vectors $Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}$ where** $\text{Span}(\vec{q}_1, \vec{q}_2) = \mathbb{R}^2$ *(HINT: What vectors do we typically append for basis extension?)*

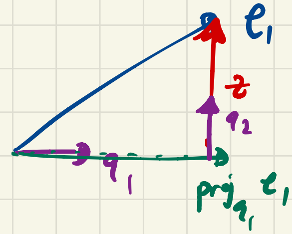



want: orthonormal basis for $\mathbb{R}^2$

$$\{v_1, e_1, e_2\} \qquad e_1 = \begin{bmatrix}1\\0\end{bmatrix} \quad e_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\|v_1\| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$



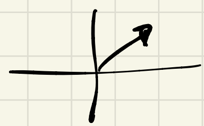

$$e_1 = \text{proj}_{q_1} e_1 + z$$

$$e_1 - \text{proj}_{q_1} e_1 = z$$

$$\begin{bmatrix}1\\0\end{bmatrix} - \frac{\langle q_1, e_1\rangle}{\cancelto{1}{\langle q_1, q_1\rangle}}\, q_1 = z$$

$$\begin{bmatrix}1\\0\end{bmatrix} - \underbrace{\langle q_1, e_1\rangle}\, q_1 = z$$

$$\frac{1}{\sqrt{2}}\underset{1\times 2}{\begin{bmatrix}1 & -1\end{bmatrix}}\underset{2\times 1}{\begin{bmatrix}1\\0\end{bmatrix}} = \frac{1}{\sqrt{2}}$$



$$z = \begin{bmatrix}1\\0\end{bmatrix} - \left(\frac{1}{\sqrt{2}}\right) q_1 = \begin{bmatrix}1\\0\end{bmatrix} - \left(\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\end{bmatrix}$$

$$q_2 = \frac{z}{\|z\|} = \begin{bmatrix}\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

want $q_2$ $\quad \{q_1, q_2\}$

(c) Now that we have calculated some $Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}$, let's apply this transformation to our original matrix $A$. **Calculate $Q^\top AQ$ and comment on the resulting matrix.**

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$Q^\top A Q = T = \begin{bmatrix} -3 & 10 \\ 0 & -3 \end{bmatrix}$$

(d) **How do the eigenvalues of the original $A$ matrix connect to the upper triangular matrix $T = Q^\top AQ$ that we calculated in the previous part.**

$$\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \qquad \frac{dx_1}{dt} = -3x_1 \qquad \frac{dx_2}{dt} = -3x_2$$

(e) Let's say you were given a system:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = \begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix} \vec{x}(t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

coupled OR decoupled ?

**Describe how you could solve for $\vec{x}(t)$ given an initial condition $\vec{x}(0)$.**

$$\frac{dx_1}{dt} = -8x_1 - 5x_2$$

have: $\frac{dx}{dt} = Ax$ want: $\frac{d\tilde{x}}{dt} = T\tilde{x}$

$$x = Q\tilde{x}$$

$$\frac{d\tilde{x}}{dt} = \begin{bmatrix} -3 & 10 \\ 0 & -3 \end{bmatrix} \tilde{x}$$

$$\frac{d}{dt} Q\tilde{x} = AQ\tilde{x}$$

$$Q\frac{d}{dt}\tilde{x} = AQ\tilde{x}$$

Q is orthonormal

$Q^{-1} = Q^T$

$$\frac{d}{dt}\tilde{x} = Q^{-1}AQ\tilde{x}$$

$$\frac{d}{dt}\tilde{x} = Q^{T}AQ\tilde{x}$$

$$\frac{d}{dt}\tilde{x} = T\tilde{x}$$

$$x(t) = Q\tilde{x}(t)$$

$$\frac{d\tilde{x}}{dt} = \begin{bmatrix} -3 & 10 \\ 0 & -3 \end{bmatrix} \tilde{x} \xrightarrow{\text{scaralize}} \begin{aligned} \frac{d}{dt}\tilde{x}_1 &= -3\tilde{x}_1 + 10\tilde{x}_2 \\ \frac{d}{dt}\tilde{x}_2 &= -3\tilde{x}_2 \end{aligned}$$

$$\begin{bmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{bmatrix} = Q^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\tilde{x}_2(t) = \tilde{x}_2(0)\, e^{-3t}$$

$$\frac{d}{dt}\tilde{x}_1 = -3\tilde{x}_1 + \underbrace{10\left[\tilde{x}_2(0)\, e^{-3t}\right]}_{u(t)}$$

$$\frac{d}{dt}\tilde{x}_1 = -3\tilde{x}_1 + u(t)$$

have $\tilde{x}_1(t)$

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$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}$$

CROWD

$$x(t) = Q\tilde{x}(t)$$

$$x(0) = Q\tilde{x}(0)$$

$$\tilde{x}(0) = Q^{-1}x(0)$$

$$\begin{bmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{bmatrix} = Q^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$