Agenda:

- Regrade Requests - 4/7
- HW 10 due 4/7
- Upper Triangularization
- Worksheet

Motivation for Upper Triangularization

- Recall: Diagonalization

Given a matrix A, we can decompose

$$A = V \Lambda V^{-1} \quad (\text{equivalently } \Lambda = V^{-1} A V)$$

where V is a matrix of eigenvectors of A
Λ is diagonal matrix of eigenvectors of A

☆ can only use if A has linearly independent eigenvectors

If we have some system & want to solve for state trajectories/stability

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$$

$\Downarrow$ diagonalize

$$\tilde{\vec{x}}[i+1] = \Lambda \tilde{\vec{x}}[i] + V^{-1}\vec{b}u[i]$$

→ decompose into independent scalar equations!

Apply scalar state trajectory, scalar condition for BIBO stability & combine to get vector case

What if we cannot diagonalize A?

→ Upper Triangular Matrix

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$$

(upper right triangle is non-zero)

Important characteristics ☆☆

- Eigenvalues still along diagonal $(t_{11}, t_{22}, \dots t_{nn})$
- Last row decoupled from rest
  → can solve for x_n then use to solve for x_{n-1} then $x_{n-2} \dots x_1$
- Matrices don't need linearly independent eigenvalues to be upper triangularized (more general than diagonalization)

Ex: Solving UT system:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} t_{11} & t_{21} & \cdots & t_{1n} \\ & t_{22} & & \vdots \\ 0 & & \ddots & \\ & & & t_{nn} \end{bmatrix} \vec{x}(t)$$

(last) nth row

$$\frac{d}{dt}x_n(t) = t_{nn}x_n(t) \quad (\text{can solve!})$$

$$\Rightarrow x_n(t) = x_n(0)e^{t_{nn} \cdot t}$$

Substitute into (n-1)th row:

$$\frac{d}{dt}x_{n-1}(t) = t_{n-1,n-1}x_{n-1}(t) + t_{n-1,n}x_n(t)$$

substitute solution for $x_n(t)$

$$\frac{d}{dt}x_{n-1}(t) = t_{n-1,n-1}x_{n-1}(t) + t_{n-1,n}\left(x_n(0)e^{t_{n,n} \cdot t}\right)$$

⇒ non-homogeneous 1st order scalar diff eq! (solvable)

⋮

Continue to back substitute until $x_1(t)$

Schur Decomposition

Theorem: ☆☆

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix w/ real λ_i.
→ Then, exists orthonormal change-of-basis $U \in \mathbb{R}^{n\times n}$ & upper triangle matrix $T \in \mathbb{R}^{n \times n}$ s.t.

$$A = UTU^T \iff T = U^T A U$$

Derivation:

- Given matrix A w/ eigenvalue λ_1 & normalized eigenvector $\vec{v}_1$
-

$$Q^TAQ = [\vec{v}_1\ R]^T[A][\vec{v}_1\ R]$$

$$= \begin{bmatrix} \vec{v}_1^T \\ R^T \end{bmatrix} [A][\vec{v}_1\ R]$$

$$= \begin{bmatrix} \vec{v}_1^T \\ R^T \end{bmatrix} [A\vec{v}_1\ AR]$$

$$= \begin{bmatrix} \vec{v}_1^T A\vec{v}_1 & \vec{v}_1 AR \\ R^T A \vec{v}_1 & R^T AR \end{bmatrix}$$

$A\vec{v}_1 = \lambda\vec{v}_1$

$$= \begin{bmatrix} \vec{v}_1^T \lambda \vec{v}_1 & \vec{v}_1 AR \\ R^T \lambda \vec{v}_1 & R^T AR \end{bmatrix}$$

$$= \begin{bmatrix} \lambda(\vec{v}_1^T\vec{v}_1) & \vec{v}_1 AR \\ \lambda(R^T\vec{v}_1) & R^T AR \end{bmatrix}$$

- $\vec{v}_1^T\vec{v}_1 = \langle \vec{v}_1, \vec{v}_1 \rangle = \|\vec{v}_1\|^2 = 1$ (normalized eigenvector)
- $R^T\vec{v}_1 = \begin{bmatrix} \text{—}\ \vec{r}_2^T\ \text{—} \\ \vdots \\ \text{—}\ \vec{r}_n^T\ \text{—} \end{bmatrix} \vec{v}_1 = \begin{bmatrix} \vec{r}_2^T\vec{v}_1 \\ \vdots \\ \vec{r}_n^T\vec{v}_1 \end{bmatrix} = \begin{bmatrix} \langle \vec{v}_1, \vec{r}_2 \rangle \\ \vdots \\ \langle \vec{v}_1, \vec{r}_n \rangle \end{bmatrix} = \vec{0}_{n \times 1}$

R is extended orthonormal basis from GS
$\Rightarrow \langle \vec{v}_1, \vec{r}_i \rangle = 0$

$$= \begin{bmatrix} \lambda_1 & \text{—}\ \vec{v}_1^T AR\ \text{—} \\ | & \\ \vec{0} & R^TAR \\ | & \end{bmatrix} \qquad \begin{bmatrix} 1\times 1 & 1 \times (n-1) \\ (n-1) \times 1 & (n-1)\times(n-1) \end{bmatrix}$$

↑
first column is "upper triangularized"
→ if R^TAR is upper triangular then we can fully upper triangularize A

☆☆ Repeat process on A

(1) Find eigenvalue of A (λ) & normalized eigenvector $(\vec{v})$
(2) Extend orthonormal basis Q from normalized eigenvector $\vec{v}$ via Gram Schmidt
(3) Calculate Q^TAQ to UT the first column & get a smaller matrix $\tilde{A}_{(n-1)\times(n-1)}$
(4) Repeat previous steps on $\hat{A}$ until reach a 1×1 matrix

Think recursion from 61A

- Base Case: 1×1 matrix is already upper triangular
- Reduce to subproblem: Apply Q to A to UT the first column
- Recursive step: UT the $(n-1)\times(n-1)$ matrix