Discussion 11A

1. Minimum Energy Control & Spectral Theorem

In controllability/reachability analysis, we try to solve the linear system:

$$\mathcal{C}_{i^\star} \begin{bmatrix} \vec{u}[0] \\ \vdots \\ \vec{u}[i^\star - 1] \end{bmatrix} = \vec{x}^\star - A^{i^\star} \vec{x}_0 \tag{1}$$

for the vector quantities $\vec{u}[0], \ldots, \vec{u}[i^\star - 1]$. Cleaning up notation, let us fix $i^\star$, let $C := \mathcal{C}_{i^\star}$, let $\vec{z} := \vec{x}^\star - A^{i^\star}\vec{x}_0$, and let $\vec{u} := \begin{bmatrix} \vec{u}[0] \\ \vdots \\ \vec{u}[i^\star - 1] \end{bmatrix}$. Then this linear system becomes

$$C\vec{u} = \vec{z} \tag{2}$$

In the real world, we would like to use this framework to control mechanical systems, often expending the **minimum energy** possible.

But what do we mean by the "energy" of the control? In this context, we will use the squared norm of the input vector $\|u\|^2 = u_1^2 + \cdots + u_n^2$ as the model for the cost to applying a set of controls to our system.

Why do we use this definition? This formula for the cost model is closely connected to many physical scenarios relating to energy. Consider a few examples below:

- $E_{\text{capacitor}} = \frac{1}{2}CV^2$
- $E_{\text{spring}} = \frac{1}{2}kx^2$
- $E_{\text{kinetic}} = \frac{1}{2}mv^2$

And so we find that the definition we use is a natural one.

Optional EECS16A Refresher: Recall the following vector spaces:

The range (or column space) of a matrix A refers to the following vector space $\mathrm{Col}(A) = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$. *It is the vector space consisting of all possible linear combinations of the columns of A.*

Then, there is the null space of A, which refers to the following vector space $\mathrm{Null}(A) = \{\vec{x} : A\vec{x} = 0\}$.

(a) Suppose we would like to solve for the minimum energy (or minimum norm) solution of the linear system $C\vec{u} = z$. This problem can be expressed as the following optimization problem:

$$\underset{\vec{u}}{\text{argmin}} \|\vec{u}\|^2 = \underset{u[i]}{\text{argmin}} \sum_{i=0}^{\ell-1} u[i]^2 \tag{3}$$

$$\text{s.t.} \quad C\vec{u} = \vec{z} \tag{4}$$

Symmetric matrices can make calculations and reasoning about the properties of a matrix much easier. Suppose C is a *real, symmetric matrix*. **Rewrite C in terms of its spectral decomposition** (take Q to be the orthonormal basis of eigenvectors of C and Λ to be the diagonal matrix of the eigenvalues).

(b) One way to look at minimum energy control is through lens of our vector spaces. How might we do this?

Well, Spectral Theorem tells us that Q is an orthonormal basis of $\mathbb{R}^n$. If $\text{Rank}(C) = r$, then Q can be written as the block matrix $\begin{bmatrix} Q_r & Q_{n-r} \end{bmatrix}$ where Q_r forms an orthonormal basis for $\text{Col}(C)$ and Q_{n-r} similarly forms one for $\text{Null}(C)$.

Let's perform an orthonormal basis change:

$$\vec{u} = Q\widetilde{\vec{u}} \tag{5}$$

Using our new basis, rewrite $\vec{u}$ in terms of Q_r and Q_{n-r}.

(HINT: Consider breaking up Q and $\vec{u}$ into a block matrix and partitioned vector respectively.)

(c) Ultimately, the objective we are trying to minimize is still $\|\vec{u}\|^2$. **Use your findings from part (b) to show that** $\|\vec{u}\|^2 = \|\widetilde{\vec{u}}_{\text{Col}(C)}\|^2 + \|\widetilde{\vec{u}}_{\text{Null}(C)}\|^2$.

(HINT: Given some arbitrary orthonormal matrix U and arbitrary vector $\vec{p}$, how are $\|\vec{p}\|$ and $\|U\vec{p}\|$ related?)

(d) Putting everything from the last two parts together, let us now solve for the solution to the minimum energy problem (written once more for convenience):

$$\underset{\vec{u}}{\text{argmin}}\,\|\vec{u}\|^2 = \underset{u[i]}{\text{argmin}} \sum_{i=0}^{\ell-1} u[i]^2 \tag{6}$$

$$\text{s.t.} \quad C\vec{u} = \vec{z} \tag{7}$$

Solve for the optimal minimum energy input $\vec{u}^*$ in its simplest form in terms of $\vec{u}_{\text{Col}(C)}$ and/or $\vec{u}_{\text{Null}(C)}$. Explain what your result means intuitively.

(HINT: Which of $\vec{u}_{\text{Col}(C)}$ or $\vec{u}_{\text{Null}(C)}$ doesn't effect $C\vec{u}$ (try to think about your vector space definitions)? What should we do to that value if we want to minimize the squared norm of the input?)

(e) Now, let's do a numerical example. Consider the following linear discrete time system

$$\vec{x}[i+1] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u[i] \tag{8}$$

Find the controllability matrix C for this system.

(f) Now, suppose we want to achieve desired state of $\vec{x}[2] = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$ at timestep $i = 1$. Assume your initial condition is $\vec{x}[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Write your linear system to solve for the input vector $\vec{u} = \begin{bmatrix} \vec{u}[0] \\ \vec{u}[1] \end{bmatrix}$ in the form $C\vec{u} = \vec{z}$. Then, solve for one $\vec{u}$ that achieves the desired system state. Remember, there will be many solutions as the system is underdetermined.

(HINT: Make use of the linear system formulation that comes as a result of controllability analysis shown on page 1.)

(g) Finally, notice that $\text{Col}(C) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ and $\text{Null}(C) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

The minimum norm solution is $\vec{u}_{\text{min}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. We will compare this to another arbitrary solution $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

i **Write both $\vec{u}_{\text{min}}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ as a linear combination of the column and null space span vectors. Compare the coefficients of the null space span vector.**

ii **Compare the norms of the two solutions. Verify that $\|\vec{u}_{\text{min}}\|$ is smaller.**

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