# Discussion 11A

**1. Minimum Energy Control & Spectral Theorem**

In controllability/reachability analysis, we try to solve the linear system:

$$\mathcal{C}_{i^\star}\begin{bmatrix}\vec{u}[0]\\ \vdots \\ \vec{u}[i^\star-1]\end{bmatrix} = \vec{x}^\star - A^{i^\star}\vec{x}_0 \tag{1}$$

for the vector quantities $\vec{u}[0],\ldots,\vec{u}[i^\star-1]$. Cleaning up notation, let us fix $i^\star$, let $C := \mathcal{C}_{i^\star}$, let $\vec{z} := \vec{x}^\star - A^{i^\star}\vec{x}_0$, and let $\vec{u} := \begin{bmatrix}\vec{u}[0]\\ \vdots \\ \vec{u}[i^\star-1]\end{bmatrix}$. Then this linear system becomes

$$C\vec{u} = \vec{z} \tag{2}$$

In the real world, we would like to use this framework to control mechanical systems, often expending the **minimum energy** possible.

But what do we mean by the "energy" of the control? In this context, we will use the squared norm of the input vector $\|u\|^2 = u_1^2 + \cdots + u_n^2$ as the model for the cost to applying a set of controls to our system.

Why do we use this definition? This formula for the cost model is closely connected to many physical scenarios relating to energy. Consider a few examples below:

- $E_{\text{capacitor}} = \frac{1}{2}CV^2$
- $E_{\text{spring}} = \frac{1}{2}kx^2$
- $E_{\text{kinetic}} = \frac{1}{2}mv^2$

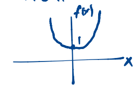
And so we find that the definition we use is a natural one.

*Optional EECS16A Refresher: Recall the following vector spaces:*

*The range (or column space) of a matrix $A$ refers to the following vector space* $\text{Col}(A) = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$*. It is the vector space consisting of all possible linear combinations of the columns of $A$.*

*Then, there is the null space of $A$, which refers to the following vector space* $\text{Null}(A) = \{\vec{x} : A\vec{x} = 0\}$.

$\min_{x \in \mathbb{R}} x^2+1 = 1$ $\quad$ $\operatorname{argmin}_{x \in \mathbb{R}} (x^2+1) = 0$






(a) Suppose we would like to solve for the minimum energy (or minimum norm) solution of the linear system $C\vec{u} = z$. This problem can be expressed as the following optimization problem:

"find the min energy sol"

$$\underset{\vec{u}}{\operatorname{argmin}} \|\vec{u}\|^2 = \underset{u[i]}{\operatorname{argmin}} \sum_{i=0}^{\ell-1} u[i]^2 \tag{3}$$

$$\text{s.t.} \quad C\vec{u} = \vec{z} \tag{4}$$

constraint

Symmetric matrices can make calculations and reasoning about the properties of a matrix much easier. Suppose $C$ is a *real, symmetric matrix*. **Rewrite $C$ in terms of its spectral decomposition** (take $Q$ to be the orthonormal basis of eigenvectors of $C$ and $\Lambda$ to be the diagonal matrix of the eigenvalues).

if $C = C^T$, then $C = Q \Lambda Q^T$

(b) One way to look at minimum energy control is through lens of our vector spaces. How might we do this?

Well, Spectral Theorem tells us that $Q$ is an orthonormal basis of $\mathbb{R}^n$. If $\text{Rank}(C) = r$, then $Q$ can be written as the block matrix $\begin{bmatrix} Q_r & Q_{n-r} \end{bmatrix}$ where $Q_r$ forms an orthonormal basis for $\text{Col}(C)$ and $Q_{n-r}$ similarly forms one for $\text{Null}(C)$.

Let's perform an orthonormal basis change:

$$\vec{u} = Q\widetilde{\vec{u}} \tag{5}$$

**Using our new basis, rewrite $\vec{u}$ in terms of $Q_r$ and $Q_{n-r}$.**

*(HINT: Consider breaking up $Q$ and $\vec{u}$ into a block matrix and partitioned vector respectively.)*

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$C \in \mathbb{R}^{n \times n}$

$\text{rank}(C) = r$

$Q \in \mathbb{R}^{n \times n}$

$$Q = \begin{bmatrix} \underbrace{\overbrace{\begin{pmatrix} | & | & | & \cdots \end{pmatrix}}^{r}}_{Q_r} & \cdots & \underbrace{\begin{pmatrix} \cdots & | & | \end{pmatrix}}_{Q_{n-r}} \end{bmatrix}$$

$\text{Col}(Q_r) = \text{Col}(C)$

$\cancel{\text{Null}(Q_{n-r}) = \text{Null}(C)}$ OR $\text{Col}(Q_{n-r}) = \text{Null}(C)$

$$u = Q\tilde{u}$$

$$u = \begin{bmatrix} Q_r & Q_{n-r} \end{bmatrix} \tilde{u}$$

$$u = \begin{bmatrix} Q_r & Q_{n-r} \end{bmatrix} \begin{bmatrix} \tilde{u}_{Col(C)} \\ \tilde{u}_{Null(C)} \end{bmatrix} = Q_r \tilde{u}_{Col(C)} + Q_{n-r} \tilde{u}_{Null(C)}$$

is $Q_r \tilde{u}_{Col(C)} \in Col(C)$ or $\cancel{\tilde{u}_{Col(C)} \in Col(C)}$

is $Q_{n-r} \tilde{u}_{Null(C)} \in Null(C)$ or $\cancel{u_{Null(C)} \in Null(C)}$?

$$Col(C) \in \mathbb{R}^n$$

$$\dim Col(C) = r \leq n$$

(c) Ultimately, the objective we are trying to minimize is still $\|\vec{u}\|^2$. **Use your findings from part (b) to show that** $\|\vec{u}\|^2 = \|\widetilde{\vec{u}}_{\mathrm{Col}(C)}\|^2 + \|\widetilde{\vec{u}}_{\mathrm{Null}(C)}\|^2$.

*(HINT: Given some arbitrary orthonormal matrix $U$ and arbitrary vector $\vec{p}$, how are $\|\vec{p}\|$ and $\|U\vec{p}\|$ related?)*

given $\|u\|^2$ how do we introduce variable $\tilde{u}$?

$u = Q\tilde{u}$

$\|u\|^2 = \|Q\tilde{u}\|^2 = \|\tilde{u}\|^2$ $\quad \|x\|^2 = x^T x$

orthonormal matrix

$(Q\tilde{u})^T(Q\tilde{u}) = \tilde{u}^T \underline{Q^T Q} \tilde{u}$
$= \tilde{u}^T \tilde{u}$

$\|\tilde{u}\|^2 \longrightarrow \|\tilde{u}_{\mathrm{Col}(C)}\|^2 + \|\tilde{u}_{\mathrm{Null}(C)}\|^2$

$$\tilde{u} = \begin{bmatrix} \tilde{u}_{\mathrm{Col}(C)} \\ \\ \tilde{u}_{\mathrm{Null}(C)} \end{bmatrix}$$

(d) Putting everything from the last two parts together, let us now solve for the solution to the minimum energy problem (written once more for convenience):

$$\underset{\vec{u}}{\operatorname{argmin}} \|\vec{u}\|^2 = \underset{u[i]}{\operatorname{argmin}} \sum_{i=0}^{\ell-1} u[i]^2 \tag{6}$$

$$\text{s.t.} \quad C\vec{u} = \vec{z} \tag{7}$$

**Solve for the optimal minimum energy input $\vec{u}^*$ in its simplest form in terms of $\vec{u}_{\mathrm{Col}(C)}$ and/or $\vec{u}_{\mathrm{Null}(C)}$. Explain what your result means intuitively.**

*(HINT: Which of $\vec{u}_{\mathrm{Col}(C)}$ or $\vec{u}_{\mathrm{Null}(C)}$ doesn't effect $C\vec{u}$ (try to think about your vector space definitions)? What should we do to that value if we want to minimize the squared norm of the input?)*

① $u = Q_r \tilde{u}_{col(C)} + Q_{n-r} \tilde{u}_{NULL(C)}$

$col(Q_r) = col(C)$

$col(Q_{n-r}) = Null(C)$

② $\|u\|^2 = \|\tilde{u}_{col(C)}\|^2 + \|\tilde{u}_{NULL(C)}\|^2$

how does $u = Q\tilde{u}$ affect $Cu = z$?

$u = Q_r \tilde{u}_{col(C)} + Q_{n-r} \tilde{u}_{NULL(C)}$

$Cu = C(Q_r \tilde{u}_{col(C)} + Q_{n-r} \tilde{u}_{NULL(C)})$

$= C Q_r \tilde{u}_{col(C)} + \underbrace{\cancel{C Q_{n-r} \tilde{u}_{NULL(C)}}}_{\vec{0}}$

$Q_{n-r}\tilde{u}_{NULL(C)} \in Null(C)$

$Cu = C Q_r \tilde{u}_{col(C)} = z$

does $C Q_r \tilde{u}_{col(C)}$ depend on $\tilde{u}_{NULL(C)}$?

$$\tilde{u} = \begin{bmatrix} \tilde{u}_{col(C)} \\ \tilde{u}_{NULL(C)} \end{bmatrix}$$

$$\arg\min_{u} \quad \|u\|^2$$

$$\text{s.t.} \quad Cu = z$$

$\tilde{u}$ coordinates

$$\arg\min_{\tilde{u}} \quad \|\tilde{u}_{col(C)}\|^2 + \|\tilde{u}_{Null(C)}\|^2$$

$$C Q_r \tilde{u}_{col(C)} = z$$

need to set $\tilde{u}_{Null(C)} = 0$

$$X = \begin{bmatrix} \boxed{x_1} \\ \boxed{\begin{matrix} x_2 \\ x_3 \\ x_4 \end{matrix}} \end{bmatrix} \begin{array}{l} \leftarrow x_{top} \\ \leftarrow x_{bottom} \end{array}$$

$$\| x \|^2 = \underbrace{(x_1^2)}_{\| x_{top} \|^2} + \underbrace{(x_2^2 + x_3^2 + x_4^2)}_{\| x_{bottom} \|^2}$$

(e) Now, let's do a numerical example. Consider the following linear discrete time system

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{A} \vec{x}[i] + \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{b} u[i] \tag{8}$$

**Find the controllability matrix $C$ for this system.**

$$C = \begin{bmatrix} | & | \\ Ab & b \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(f) Now, suppose we want to achieve desired state of $\vec{x}[2] = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$ at timestep $i = 2$. Assume your initial condition is $\vec{x}[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

**Write your linear system to solve for the input vector $\vec{u} = \begin{bmatrix} \vec{u}[0] \\ \vec{u}[1] \end{bmatrix}$ in the form $C\vec{u} = \vec{z}$. Then, solve for one $\vec{u}$ that achieves the desired system state.** Remember, there will be many solutions as the system is underdetermined.

*(HINT: Make use of the linear system formulation that comes as a result of controllability analysis shown on page 1.)*

$$x[1] = Ax[0] + bu[0]$$

$$x[2] = Ax[1] + bu[1]$$

$$x[2] = A^2 x[0] + Abu[0] + bu[1]$$

$$x[2] = Abu[0] + bu[1]$$

$$x[2] = \begin{bmatrix} | & | \\ Ab & b \\ | & | \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \end{bmatrix}$$

$$\vec{z} = x[2] = Cu$$

$$\begin{bmatrix} 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \end{bmatrix} \qquad u_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

  

(g) Finally, notice that $\text{Col}(C) = \text{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$ and $\text{Null}(C) = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$. $C = \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$

The minimum norm solution is $\vec{u}_{\text{min}} = \begin{bmatrix}2\\-2\end{bmatrix}$. We will compare this to another arbitrary solution $\begin{bmatrix}3\\-1\end{bmatrix}$.

i **Write both $\vec{u}_{\text{min}}$ and $\begin{bmatrix}3\\-1\end{bmatrix}$ as a linear combination of the column and null space span vectors. Compare the coefficients of the null space span vector.**

$$\begin{bmatrix}2\\-2\end{bmatrix} = \underline{2}\begin{bmatrix}1\\-1\end{bmatrix} + \cancel{\underline{0}\begin{bmatrix}1\\1\end{bmatrix}} = 0$$

ii **Compare the norms of the two solutions. Verify that $\|\vec{u}_{\text{min}}\|$ is smaller.**

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