Agenda:

- HW 11 - due 4/14
- Minimum Energy Control
- Spectral Theorem

Minimum Energy Control

Motivation:

Recall Controllability/Reachability:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$$

state trajectory $\left\{ \vec{x}[i] = A^i \vec{x}_0 + \sum_{j=0}^{i-1} A^{i-1-j} B \vec{u}[j] \right.$

$$\vec{x}[i] = A^i \vec{x}_0 + C_i \begin{bmatrix} \vec{u}[0] \\ \vdots \\ \vec{u}[i-1] \end{bmatrix}$$

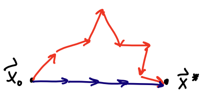
$$\Rightarrow C_{i^*} \begin{bmatrix} \vec{u}[0] \\ \vdots \\ \vec{u}[i^*-1] \end{bmatrix} = \vec{x}^* - A^{i^*}\vec{x}_0$$

Reachable if $(\vec{x}^* - A^{i^*}\vec{x}_0) \in Col(C_{i^*})$

→ could exist many/infinite solutions

Q: Which solution is best?

Goal: Pick solution that minimizes energy



suboptimal path

optimal (min energy) path

Def: Energy

$$\vec{w} = \begin{bmatrix} \vec{u}[0] \\ \vdots \\ \vec{u}[i^*-1] \end{bmatrix} \longrightarrow \|\vec{w}\|^2 = \sum_{i=0}^{i^*-1} \|\vec{u}[i]\|^2$$

- squared norm

Optimization Problem:

$$\min_{\vec{w}} \|\vec{w}\|^2 \quad \text{s.t.} \quad C\vec{w} = \vec{z}$$

Main Idea: want to choose some $\vec{w}$ that has no component in the Null(C) b/c this won't help us reach our goal but will unnecessarily increase the energy of our solution

→ covered in Dis 11A worksheet

Spectral Theorem

☆☆ Let $A \in \mathbb{R}^{n \times n}$ be real & symmetric

1. eigenvalues are real
2. A is diagonalizable
3. Exists orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A

⇒ A may be "orthonormally diagonalized"

$$A = V \Lambda V^T \Leftrightarrow \Lambda = V^T A V$$

where Λ: diagonal matrix of eigenvalues
V: orthonormal matrix of eigenvectors of A

Motivation: symmetric matrices have some really nice properties that we can exploit

→ will learn how to generalize to non-symmetric/non-square matrices (SVD)

Proofs for Spectral Theorem (not as important but should still be familiar)

② Diagonalizable

$S = UTU^T$ by Schur Decomposition

$$S^T = (UTU^T)^T = (U^T)^T T^T U^T = UT^TU^T$$

By symmetry:

$$S = S^T$$

$$\Rightarrow UTU^T = UT^TU^T$$

$$\Rightarrow T = T^T$$

$$\begin{bmatrix} \diagdown & \\ 0 & \end{bmatrix} = \begin{bmatrix} \diagdown & 0 \\ & \end{bmatrix} = \begin{bmatrix} \diagdown & 0 \\ 0 & \end{bmatrix}$$

③ Diagonalizable by orthonormal basis of eigenvectors

$S = UDU^T$ } right multiply by U

$SU = UD$

$$S[\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_n] = [\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_n] \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \ d_n \end{bmatrix}$$

(1) $S\vec{u}_1 = d_1\vec{u}_1$
(2) $S\vec{u}_2 = d_2\vec{u}_2$
⋮
(n) $S\vec{u}_n = d_n\vec{u}_n$

By def of eigenvalue/eigenvectors
$d_1 \ldots d_n$ are eigenvalues
$\vec{u}_1 \ldots \vec{u}_n$ are (orthonormal) eigenvectors

① Real Eigenvalues

$$S\vec{v} = \lambda\vec{v}$$

Let $\lambda = \lambda_r + j \cdot \lambda_i$ ⇒ Show $\lambda = \lambda^*$

(a) $S\vec{v} = \lambda\vec{v}$

$$\vec{v}^*(S\vec{v}) = \vec{v}^*\lambda\vec{v}$$

$$\vec{v}^* S \vec{v} = \lambda(\vec{v}^*\vec{v})$$

(b) $(S\vec{v} = \lambda\vec{v})^*$

$$\vec{v}^* S^* = \lambda^* \vec{v}^*$$

$$(\vec{v}^* S^*)\vec{v} = \lambda^* \vec{v}^* \vec{v}$$

$$\vec{v}^* S \vec{v} = \lambda^*(\vec{v}^*\vec{v})$$

$$\Rightarrow \lambda(\vec{v}^*\vec{v}) = \lambda^*(\vec{v}^*\vec{v})$$

$$\Rightarrow \lambda = \lambda^* \checkmark$$

Extra Stuff (Q1b): (unnecessary for now, but here if curious)

Q: Why are we allowed to break up

$$Q = \begin{bmatrix} Q_r & Q_{n-r} \end{bmatrix} \qquad (\text{from } C = Q\Lambda Q^T)$$

where Q_r spans the columnspace of C

Q_{n-r} spans the nullspace of C

A: By Spectral Theorem

$C = Q\Lambda Q^T$ where Q are eigenvectors of C

$\Rightarrow CQ = Q\Lambda$

Arrange eigenvalues in Λ:

$\lambda_1, \lambda_2, \dots \lambda_r \neq 0, \; \lambda_{r+1}, \lambda_{r+2}, \dots \lambda_n = 0$

$$\Rightarrow CQ = Q \begin{bmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix}$$

nonzero eigenvalues $\quad$ 0 eigenvalues

$$\Rightarrow C\left[\vec{q}_1 \cdots \vec{q}_r \; \vec{q}_{r+1} \cdots \vec{q}_n\right] = \left[\vec{q}_1 \cdots \vec{q}_r \; \vec{q}_{r+1} \cdots \vec{q}_n\right] \begin{bmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix}$$

$\Rightarrow$ Non-zero eigenvalues:

$C\vec{q}_1 = \lambda_1 \vec{q}_1$

Zero Eigenvalues $(\lambda_{r+1}, \dots \lambda_n)$

$$\left.\begin{aligned} C\vec{q}_{r+1} &= 0 \\ &\vdots \\ C\vec{q}_n &= 0 \end{aligned}\right\} \vec{q}_{r+1} \dots \vec{q}_n \in \text{Null}(C) \Rightarrow \text{form a basis for Null}(C) \;☆ \; ✓$$

Now:

$$Q = [\underbrace{\vec{q}_1 \cdots \vec{q}_r}_{\text{span Col}(C)} \mid \underbrace{\vec{q}_{r+1} \cdots \vec{q}_n}_{\text{span Null}(C) \; ✓}]$$

(also need to prove this, but will leave for SVD since it is more involved)

☆ Main Takeaway from this proof:

→ eigenvectors corresponding to 0 eigenvalues span the nullspace of a matrix