Discussion 11B

1. Computing the SVD: A "Tall" Matrix Example

$A = U \Sigma V^T$

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}. \tag{1}$$

$(3\times 2) = (3 \times 3)(3\times 2)(2 \times 2)$

Here, we expect $U \in \mathbb{R}^{3\times 3}$, $\Sigma \in \mathbb{R}^{3\times 2}$, and $V \in \mathbb{R}^{2\times 2}$ (recall that U and V must be square since they are orthonormal matrices).

In this problem, we will walk through the SVD algorithm, prove some important theorems about the SVD matrices and column/null spaces, and consider an alternate way to approach the SVD.

(a) Let's start by trying to write A as an outer product in the form of $\sigma \vec{u}\vec{v}^\top$ where both $\vec{u}$ and $\vec{v}^\top$ have unit norm. *(HINT: Are the columns of A linearly independent or dependent? What does that tell us about how we can represent them?)*

(b) In this part, we will walk through Algorithm 7 in Note 16. This algorithm applies for a general matrix $A \in \mathbb{R}^{m\times n}$.

i. **Find** $r := \text{rank}(A)$**. Compute** $A^\top A$ **and diagonalize it using the spectral theorem (i.e. find** V **and** Λ**).**

ii. **Unpack** $V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$ **and unpack** $\Lambda := \begin{bmatrix} \Lambda_r & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{bmatrix}$**.**

iii. **Find** $\Sigma_r := \Lambda_r^{1/2}$ **and then find** $\Sigma := \begin{bmatrix} \Sigma_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$**.**

iv. **Find** $U_r := AV_r\Sigma_r^{-1}$**, where** $U_r \in \mathbb{R}^{3\times 1}$ **and then extend the basis defined by columns of** U_r **to find** $U \in \mathbb{R}^{3\times 3}$**.**
(HINT: How can we extend a basis, and why is that needed here?)

$$\underset{(3\times 2)}{\begin{bmatrix} & \\ & \\ & \end{bmatrix}} = \underset{(3\times 1)}{\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}} \underset{(1\times 2)}{\begin{bmatrix} \cdot & \cdot \end{bmatrix}}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \\ u_3 v_1 & u_3 v_2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} = \begin{bmatrix} \cdot & \\ & \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}_{u} \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{v^T}$$

$$\sqrt{1^2 + (-2)^2 + (2)^2}$$

$$\sqrt{9}$$

$$\|v\| = \sqrt{2}$$

$$\|u\| = 3$$

$$\hat{u} = \frac{u}{\|u\|} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} \qquad \hat{v}^T = \frac{v^T}{\|v\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$u = \|u\| \hat{u} \qquad v = \|v\| \hat{v}$$

$$A = uv^T = (\|u\| \hat{u})(\|v\| \hat{v})^T$$

$$= \|u\| \|v\| \hat{u} \hat{v}^T = \sigma \hat{u} \hat{v}^T = 3\sqrt{2} \begin{bmatrix} \frac{1}{3} \\ -2/3 \\ 2/3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\sigma = \|u\| \|v\| = 3 \cdot \sqrt{2}$$

v. **Use the previous parts to write the full SVD of A.**

vi. **If we were to calculate the SVD of our matrix using a calculator, are we *guaranteed* to always get the same SVD? Why or why not?**

(c) We now want to create the SVD of $A^\top$. **What are the relationships between the matrices composing A and the matrices composing $A^\top$?**

$$A^T A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$(2\times3)(3\times2)$ $\qquad$ (2×2)

$r = 1$

Reminder: Spectral Theorem

$S = S^T$

$$S = Q \Lambda Q^T \qquad Q^T = Q^{-1}$$
$$= Q \Lambda Q^{-1}$$

$$\det(A^TA - \lambda I) = \det\begin{pmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{pmatrix}$$

$$= (9-\lambda)^2 - 81 = 0$$

$$(9-\lambda)^2 = 81$$

$$9-\lambda = \pm 9$$

$$9-\lambda = 9 \qquad 9-\lambda = -9$$

$$\lambda = 0, 18$$

$\underline{\underline{\lambda = 0}}$

$$\begin{bmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{bmatrix}$$

$$\text{null}\begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\lambda = 18$

$$\text{null}\begin{pmatrix} -9 & -9 \\ -9 & -9 \end{pmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

$$v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$(A^TA) = Q \Lambda Q^T$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 18 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$A^TA = (U\Sigma V^T)^T U \Sigma V^T$$
$$= (V \Sigma^T U^T) U \Sigma V^T$$
$$= V \Sigma^T \Sigma V^T$$

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

$$A^TA = V \Sigma^T \Sigma V^T \qquad A^TA = Q \Lambda Q^T$$

$$V = Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 18 & 0 \\ 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} v_r & v_{n-r} \end{bmatrix} =$$

$$\Lambda_r = 18$$

$$v_r = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad v_{n-r} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\Lambda_r = 18$

$$\Sigma_r = \Lambda_r^{1/2} = \sqrt{18}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} U_r & U_{n-r} \end{bmatrix}$$

$$V_r = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U_r = A V_r \Sigma_r^{-1}$$

$$U_r = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \frac{1}{\sqrt{18}} \qquad \Sigma_r = \sqrt{18}$$

$$= \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

use basis vectors for $\mathbb{R}^3$

$$\left\{ \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}, e_1, e_2, e_3 \right\} \longrightarrow \text{G.S.}$$

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