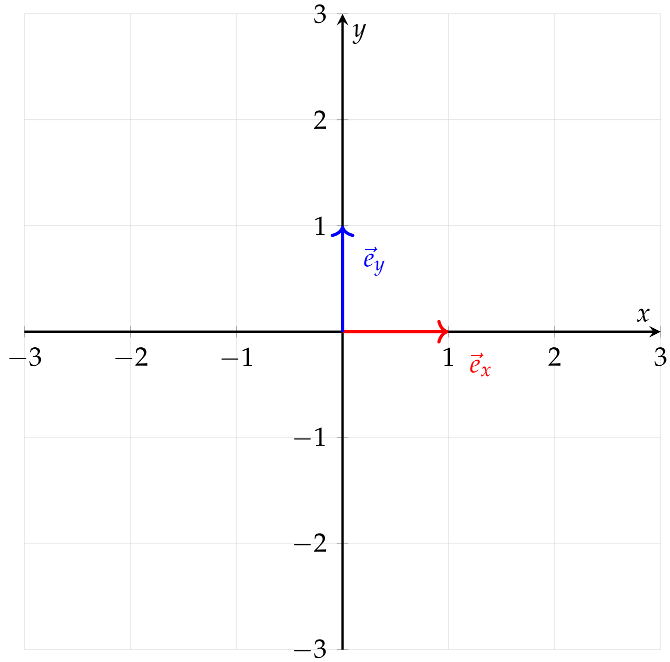
Discussion 12A

1. Geometric Interpretation of the SVD

(a) When we plot the transformation given by a specific matrix, we think about how the matrix transforms the standard basis vectors. In 2D, let $\vec{e}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The vectors $\vec{e}_x$ and $\vec{e}_y$ are shown below



Consider the following matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \tag{1}$$

How would A transform $\vec{e}_x$ and $\vec{e}_y$? Plot the result.

(b) Let's take a look at a special 2×2 matrix.

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

Show that this matrix is orthonormal. This matrix is called a rotation matrix and will rotate any vector counterclockwise by θ degrees.

(c) Now let's consider how this transformation looks in the lens of the SVD. You are given the following matrix A:

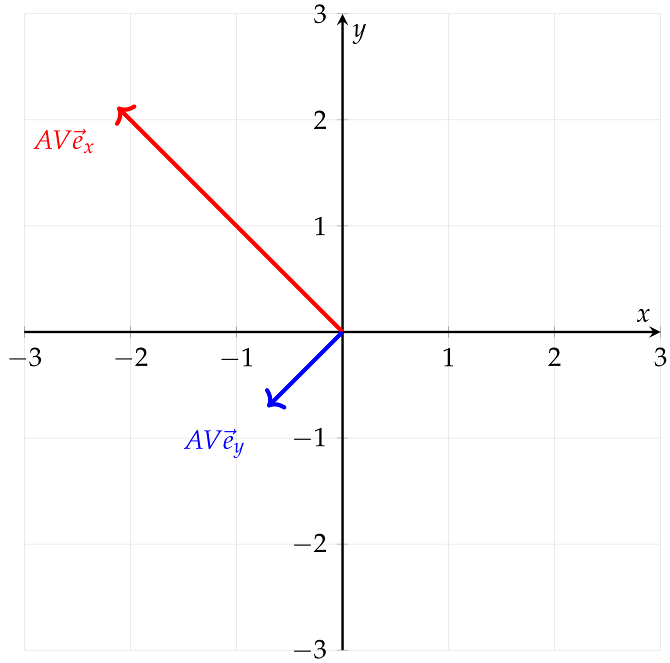
$$A = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix} \tag{3}$$

Recall that the SVD of this matrix is given by $A = U\Sigma V^\top$. Assume you are told that

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{4}$$

We will try to deduce U and Σ graphically, and the confirm our results numerically. **Plot the transformation given by V by showing how it affects $\vec{e}_x$ and $\vec{e}_y$ via left multiplication.** *(HINT: Try writing V as a rotation matrix with a specific θ.)*

(d) Suppose you are told that the transformation of AV on $\vec{e}_x$ and $\vec{e}_y$ looks like



Write this transformation AV in terms of U and Σ. Recall that the U matrix is an orthonormal matrix so it will correspond to any rotations or reflections, and the Σ matrix is a diagonal matrix and will perform any scaling operations. **Based on this fact and the plot of the transformation above, write down a guess for what U and Σ might be.**

(e) **Based on the given V matrix, compute the SVD.** Does your answer match your hypothesis from the previous part?

(f) **Using your answer for U and Σ from the previous part, plot the transformation of Σ on $\vec{e}_x$ and $\vec{e}_y$. From here, plot the transformation of $U\Sigma$ on $\vec{e}_x$ and $\vec{e}_y$.** Does the final plot resemble the transformation shown by AV?

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