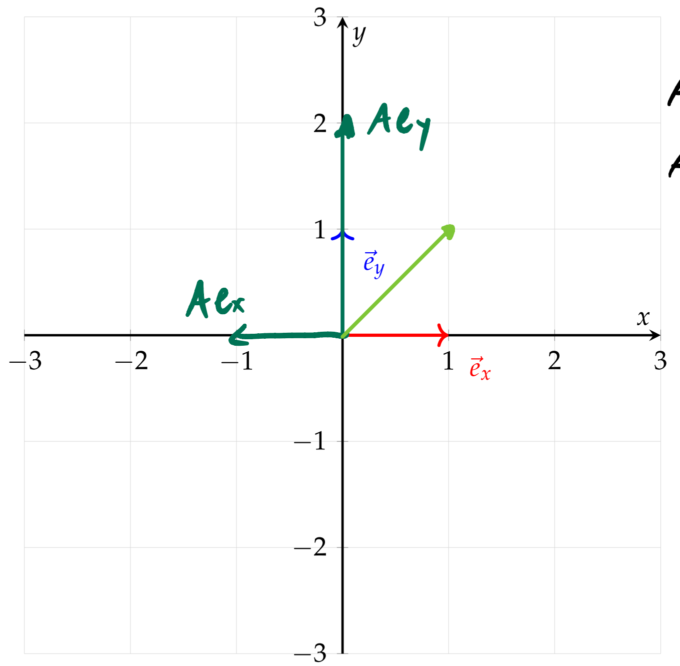
Discussion 12A

1. Geometric Interpretation of the SVD

(a) When we plot the transformation given by a specific matrix, we think about how the matrix transforms the standard basis vectors. In 2D, let $\vec{e}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The vectors $\vec{e}_x$ and $\vec{e}_y$ are shown below



$A(\alpha_1 e_x) = \alpha_1 A(e_x)$

$A(\alpha_2 e_y) = \alpha_2 A(e_y)$

Consider the following matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad (1)$$

How would A transform $\vec{e}_x$ and $\vec{e}_y$? Plot the result.

$A e_x = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$A e_y = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

description: reflection about the y-axis

description: scaling by a factor of 2.

(b) Let's take a look at a special 2×2 matrix.

$$\begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$R = \begin{array}{cc} u_1 & u_2 \\ \end{array}$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

Show that this matrix is orthonormal. This matrix is called a rotation matrix and will rotate any vector counterclockwise by θ degrees.

columns are normalized

columns are mutually orthogonal

R is square

$$\left\| \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$u_1 \cdot u_2$

$u_1^T u_2$

$$\begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$$

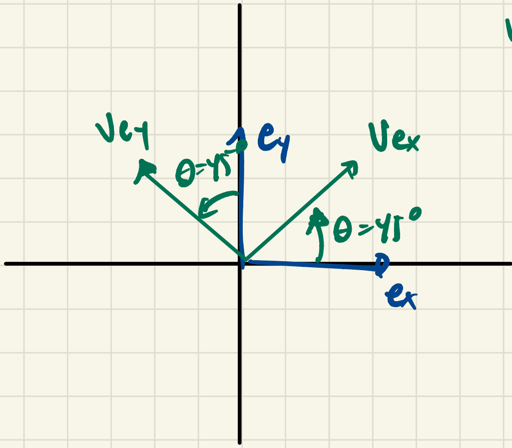
(c) Now let's consider how this transformation looks in the lens of the SVD. You are given the following matrix A:

$$A = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix} \tag{3}$$

Recall that the SVD of this matrix is given by $A = U\Sigma V^\top$. Assume you are told that

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{4}$$

We will try to deduce U and Σ graphically, and the confirm our results numerically. **Plot the transformation given by V by showing how it affects $\vec{e}_x$ and $\vec{e}_y$ via left multiplication.** *(HINT: Try writing V as a rotation matrix with a specific θ.)*



$$V_{e_x} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

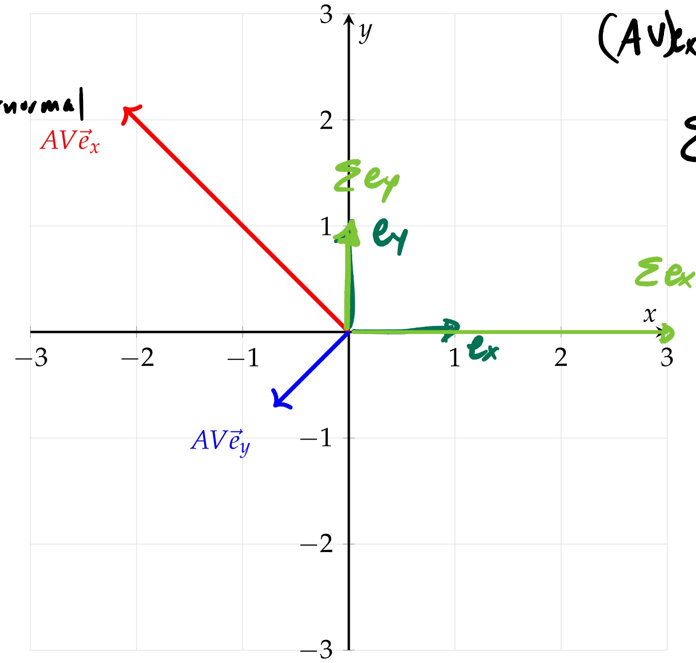
$$= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$V_{e_y} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

(d) Suppose you are told that the transformation of AV on $\vec{e}_x$ and $\vec{e}_y$ looks like

$\|Ux\| = \|x\|$

if U is orthonormal



$(AV)e_x = (U\Sigma)e_x$

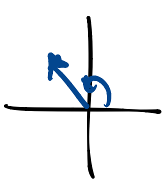
$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Write this transformation AV in terms of U and Σ. Recall that the U matrix is an orthonormal matrix so it will correspond to any rotations or reflections, and the Σ matrix is a diagonal matrix and will perform any scaling operations. **Based on this fact and the plot of the transformation above, write down a guess for what U and Σ might be.**

$$A = U\Sigma V^T$$

$$AV = (U\Sigma V^T)V = U\Sigma$$

V is orthonormal



$$AVe_x = U\Sigma e_x = U\begin{bmatrix} 3 & 0 \\ 0 & \sigma_2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$AVe_y = U\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

(e) **Based on the given V matrix, compute the SVD.** Does your answer match your hypothesis from the previous part?

(f) **Using your answer for U and Σ from the previous part, plot the transformation of Σ on $\vec{e}_x$ and $\vec{e}_y$. From here, plot the transformation of $U\Sigma$ on $\vec{e}_x$ and $\vec{e}_y$.** Does the final plot resemble the transformation shown by AV?

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given A, V find U, Σ

$$V = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$$

$$AV = U\Sigma \boxed{V^T V}$$

$$\hookrightarrow AV = U\Sigma$$

$$A \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ Av_1 & Av_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \sigma_1 u_1 & \sigma_2 u_2 \\ | & | \end{bmatrix}$$

$$Av_1 = \sigma_1 u_1$$

$$A = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix} \qquad V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$Av_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} \\ 2/\sqrt{2} + 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -3/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix} = 3 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

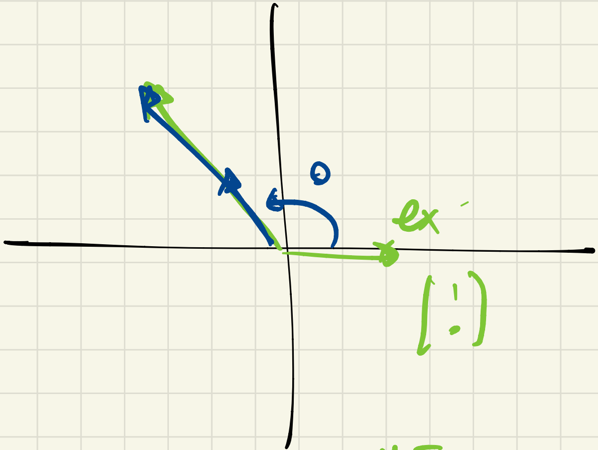
$\sigma_1 = 3$

$u_1 =$

$$Av_2 = \begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \sigma_2 u_2$$

$\sigma_2 = 1$

$$u_2 = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$U \Sigma e_x$

$$U \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\sigma_1 U \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$