Agenda:

- HW12 released - due 4/22
- 2 weeks left! or
- Course Evaluations are out!
  - $\rightarrow$ check your email
- SVD (Continued)

$\boxed{\text{Recap:}}$

Given any $A \in \mathbb{R}^{m \times n}$ w/ $\text{rank}(A) = r$

- $A = U \Sigma V^T$ (Full SVD)
- $A = U_r \Sigma_r V_r^T$ (Compact SVD)
- $A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$ (outer product form)

$\rightarrow U$ is orthonormal $\in \mathbb{R}^{m \times m}$

$$\left[\begin{array}{ccc|ccc} \vec{u}_1 & \cdots & \vec{u}_r & \vec{u}_{r+1} & \cdots & \vec{u}_m \end{array}\right] = [U_r, U_{m-r}]$$

- $U_r = [\vec{u}_1 \cdots \vec{u}_r]$ span $\text{Col}(A)$
- $U_{m-r} = [\vec{u}_{r+1} \cdots \vec{u}_m]$ span $\text{Null}(A^T)$

$\rightarrow V$ is orthonormal $\in \mathbb{R}^{n \times n}$

$$\left[\begin{array}{ccc|ccc} \vec{v}_1 & \cdots & \vec{v}_r & \vec{v}_{r+1} & \cdots & \vec{v}_n \end{array}\right] = [V_r, V_{n-r}]$$

- $V_r = [\vec{v}_1 \cdots \vec{v}_r]$ span $\text{Col}(A^T)$ (aka. rowspace)
- $V_{n-r} = [\vec{v}_{r+1} \cdots \vec{v}_n]$ span $\text{Null}(A)$

$\rightarrow \Sigma$ is "diagonal" $\in \mathbb{R}^{m \times n}$

$$\left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right] = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}$$

Solving Linear Systems using SVD

$$A\vec{x} = \vec{b}$$

① A is square, invertible

$$\vec{x} = A^{-1}\vec{b}$$

② A is tall, linear independent cols

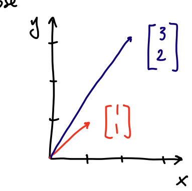
$$\vec{x} = (A^TA)^{-1}A^T\vec{b}$$

③ A is tall/wide, linear independent/dependent cols

$$\vec{x} = A^{\dagger}\vec{b}$$

$\rightarrow A^{\dagger} = V_r \Sigma_r^{-1} U_r^T$ (pseudo-inverse, coming up)

- generalizes solution b/c SVD allows us decompose any matrix into $U\Sigma V^T$



Geometric Interpretation of Matrices

① Diagonal Matrix $\Rightarrow$ scaling

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$ $\Rightarrow$ scales x by 3, y by 2

② Orthonormal Matrix $\Rightarrow$ rotation/reflection (norm remains constant)

$$\vec{y} = U\vec{x}$$

$$\|\vec{y}\|^2 = \langle \vec{y}, \vec{y} \rangle = \langle U\vec{x}, U\vec{x} \rangle = (U\vec{x})^T(U\vec{x}) = \vec{x}^TU^TU\vec{x}$$

$$= \vec{x}^T\vec{x} = \langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2$$

③ Any other matrix: combination of scaling/rotation/reflection

SVD: $U\Sigma V^T \Rightarrow$ rotation/reflection + scaling + rotation/reflection