Agenda

- HW12 due 4/22
- Course Evals
- Lab Design Competition
- Apply for Fall 2023 Course Staff
- Pseudoinverse

Comparison of SVD Forms (Note 14)

$A \in \mathbb{R}^{m\times n}$, $rank(A)=r$

		Strengths	Weaknesses
① Full SVD	$A=U\Sigma V^T$	· U,V are orthonormal (square) · $Null(A^T)$ as $Col(U_{m-r})$ · $Null(A)$ as $Col(V_{n-r})$	· computationally intensive ↳ need GS (twice) · Σ non-square, not invertible
② Compact SVD	$A=U_r\Sigma_r V_r^T$	· Σ_r is square, invertible · easier to construct than Full SVD	· U_r, V_r not square, invertible · no characterization of Null
③ Outer Product SVD	$A=\sum_{i=1}^{r}\sigma_i\vec{u}_i\vec{v}_i^T$	· computationally efficient · easier to construct than Full SVD	· summation notation messy · no characterization of Null

Projections (onto orthonormal bases)

Projection can be thought of as what least squares solution maps to or the vector closest to $\vec{b}$ in $Col(A)$.

$A\vec{x}=\vec{b}$ (A is tall, linear independent cols)

$\Rightarrow \vec{x}^* = (A^TA)^{-1}A^T\vec{b}$ (least squares solution)

$A\vec{x}^*=\vec{p}$

$\Rightarrow proj_{Col(A)}\vec{b} = A(A^TA)^{-1}A^T\vec{b}$

If A is orthonormal:

$$proj_{Col(A)}\vec{b} = AA^T\vec{b}$$

Minimum Norm Solution ☆☆

$A\vec{x}=\vec{b}$

$A\in\mathbb{R}^{m\times n}$ is wide $(m\le n)$ with full column rank.

Then, the minimum norm solution can be represented as:

$$\vec{x}^* = A^T(AA^T)^{-1}\vec{b}$$

Proof: $A=U\Sigma V^T\in\mathbb{R}^{m\times n}$ $(m\le n)$, $rank(A)=n$

$\Rightarrow$ Dis 11A: minimum norm solution is orthogonal to the $Null(A)$

$\rightarrow$ We know $Col(A^T)$ is $\perp$ to $Null(A)$

Therefore, let's represent $\vec{x}=A^T\vec{\tilde{x}}$ where $\vec{x}$ is a vector in $Col(A^T)$ rowspace

$\Rightarrow A(A^T\vec{\tilde{x}})=\vec{b}$

$\Rightarrow AA^T\vec{\tilde{x}}=\vec{b}$

$\Rightarrow (AA^T)^{-1}(AA^T)\vec{\tilde{x}}=(AA^T)^{-1}\vec{b}$

$\Rightarrow \vec{\tilde{x}}=(AA^T)^{-1}\vec{b}$

$\Rightarrow A^T\vec{\tilde{x}}=A^T(AA^T)^{-1}\vec{b}$

$$\boxed{\vec{x}=A^T(AA^T)^{-1}\vec{b}}$$

☆☆ Pseudoinverse

① Full Pseudoinverse:

$A\in\mathbb{R}^{m\times n}$, $rank(A)=r\le\min\{m,n\}$

this just means can be rank-deficient in rows & columns

$\rightarrow$ The pseudoinverse is given by:

$$A^\dagger = V\Sigma^\dagger U^T \quad \text{where} \quad \Sigma^\dagger = \left[\begin{array}{c|c}\Sigma_r & 0\\ \hline 0 & 0\end{array}\right]^\dagger = \left[\begin{array}{c|c}\Sigma_r^{-1} & 0\\ \hline 0 & 0\end{array}\right]$$

↳ dagger will invert all non-zero diagonal elements

② Compact Pseudoinverse

Given $A=U_r\Sigma_rV_r^T$, then

$$A^\dagger = V_r\Sigma_r^{-1}U_r^T$$

$\rightarrow$ this works b/c U_{m-r}, V_{n-r} will still correspond to zero singular values & therefore do not contribute to A

☆ Connecting back to Dis 10

↳ pseudo-inverse will basically disregard an vectors in the $Null(A)$

Properties of Pseudo-inverse (prove on own time ◡)

- If A is invertible, $A^{-1}=A^\dagger$
- $(A^\dagger)^\dagger = A$
- $(A^T)^\dagger = (A^\dagger)^T$
- $\alpha\neq 0$, $(\alpha A)^\dagger = \alpha^{-1}A^\dagger$
- $AA^\dagger A = A$
- $A^\dagger AA^\dagger = A^\dagger$
- $AA^\dagger = U_rU_r^T$ (projection onto $Col(A)$)
- $A^\dagger A = V_rV_r^T$ (projection onto $Col(A^T)$)

we will show these in Dis 12B