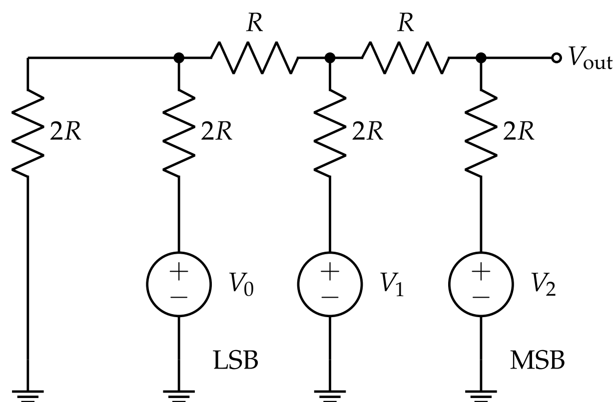
Homework 1

This homework is due on Friday, January 27, 2023, at 11:59PM. Self-grades and HW Resubmissions are due on the following Friday, Feburary 3, 2023, at 11:59PM.

1. **Digital-Analog Converter**

A digital-analog converter (DAC) is one of the key interface components between the digital and the analog world. It is a circuit for converting a digital representation of a number (binary) into a corresponding analog voltage. In this problem, we will consider a DAC made out of resistors only (resistive DAC) called the R-$2R$ ladder. This DAC will help us generate the analog voltages from the digital representation, and later will also help us digitize the analog voltages when we will be building analog to digital interfaces in Lab 3, in part based on this ladder-DAC.

Here is the circuit for a 3-bit resistive DAC.

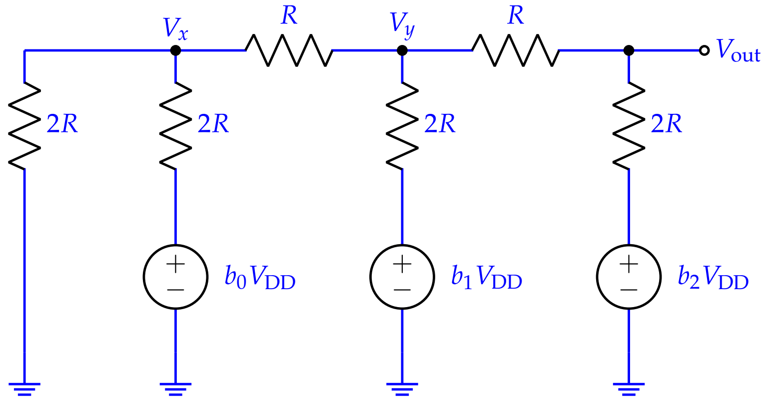


Let $b_0, b_1, b_2 = \{0, 1\}$ (that is, either 1 or 0), and let the voltage sources $V_0 = b_0 V_{DD}$, $V_1 = b_1 V_{DD}$, $V_2 = b_2 V_{DD}$, where V_{DD} is the supply voltage.

As you may have noticed, (b_2, b_1, b_0) represents a 3-bit binary (unsigned) number where each of b_i is a binary bit. b_0 is the least significant bit (LSB) and b_2 is the most significant bit (MSB). We will now analyze how this converter functions.

(a) **Solve for V_{out} in terms of V_{DD} and the binary bits b_2, b_1, b_0.**

Solution: There are several ways to solve this problem. The first way is to use KCL and create a system of equations which we solve for using Gaussian elimination.



Applying KCL at nodes V_x, V_y, and V_{out} and substituting in for the currents through the resistors, we get

$$\frac{V_x}{2R} + \frac{V_x - b_0 V_{\text{DD}}}{2R} + \frac{V_x - V_y}{R} = 0 \tag{1}$$
$$\frac{V_y - b_1 V_{\text{DD}}}{2R} + \frac{V_y - V_x}{R} + \frac{V_y - V_{\text{out}}}{R} = 0 \tag{2}$$
$$\frac{V_{\text{out}} - b_2 V_{\text{DD}}}{2R} + \frac{V_{\text{out}} - V_y}{R} = 0 \tag{3}$$

This system of equations can be solved using substitution or Gaussian elimination. One approach is shown below:

Multiplying (1), (2), (3) by R, we get

$$2V_x - \frac{b_0 V_{\text{DD}}}{2} - V_y = 0 \tag{4}$$
$$\frac{5V_y}{2} - \frac{b_1 V_{\text{DD}}}{2} - V_x - V_{\text{out}} = 0 \tag{5}$$
$$\frac{3V_{\text{out}}}{2} - \frac{b_2 V_{\text{DD}}}{2} - V_y = 0 \tag{6}$$

Adding $\frac{1}{4}\times$ (4), $\frac{1}{2}\times$ (5), and (6), we get

$$V_{\text{out}} - \frac{b_2 V_{\text{DD}}}{2} - \frac{b_1 V_{\text{DD}}}{4} - \frac{b_0 V_{\text{DD}}}{8} = 0 \tag{7}$$
$$\implies \frac{b_2 V_{\text{DD}}}{2} + \frac{b_1 V_{\text{DD}}}{4} + \frac{b_0 V_{\text{DD}}}{8} = V_{\text{out}} \tag{8}$$

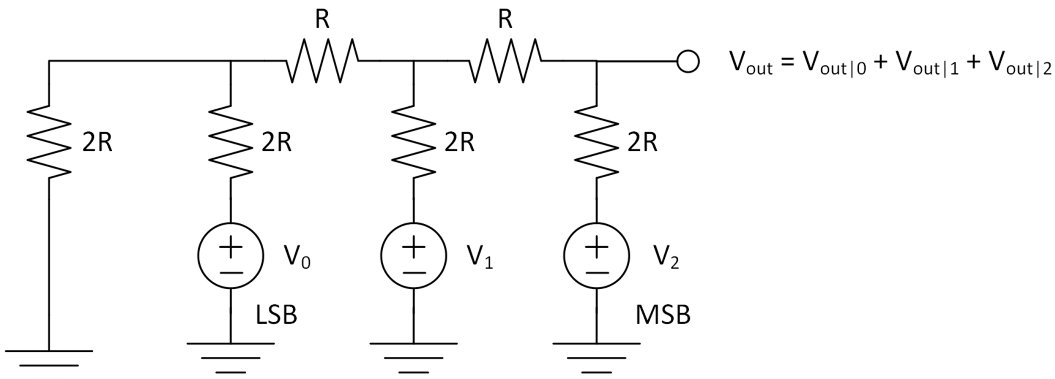
While using KCL and Gaussian elimination is a correct way of solving this problem, there is another way by using superposition and Thevenin equivalent circuits which you learned in EE16A. This approach tends to be more "intuitive": it helps you understand why the circuit was designed the way it was, how to quickly solve for the circuit (in your head, with enough practice), and transfer key design principles to invent new circuits for new problems. These skills are key to a successful career in engineering.

Recall that the output voltage is a superposition of all the independent voltage and current sources in the circuit as seen at the output. In other words, we can turn on each independent source separately, solve for the output voltage for that given source, repeat for each independent source, and sum the output voltages for each case. This will be the final output voltage.

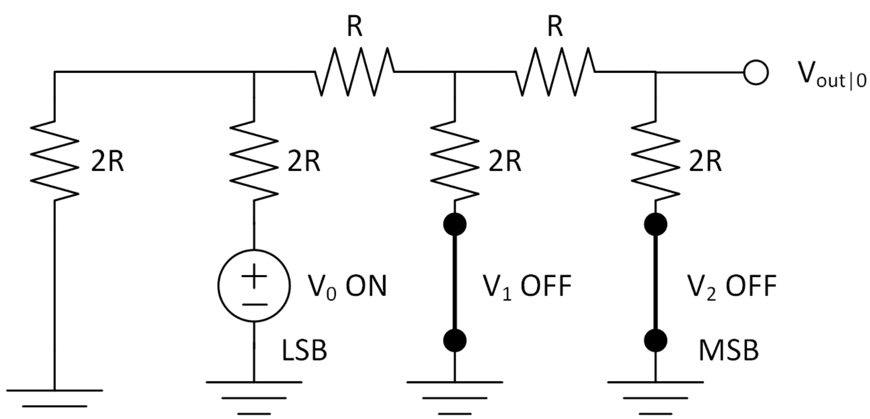
Mathematically, we can write this as:

$$V_{\text{out}} = V_{\text{out}\,|\,0} + V_{\text{out}\,|\,1} + V_{\text{out}\,|\,2} \tag{9}$$

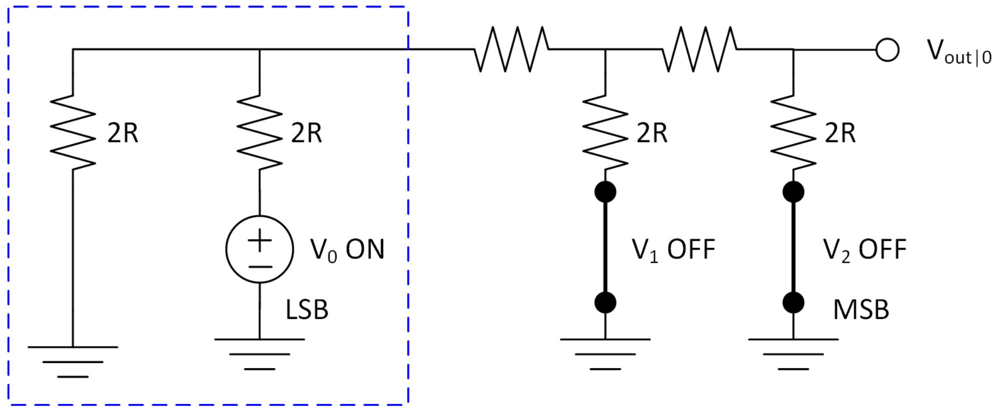
where $V_{\text{out}\,|\,0}$ refers to V_{out} due to independent source 0 on and all other sources off.

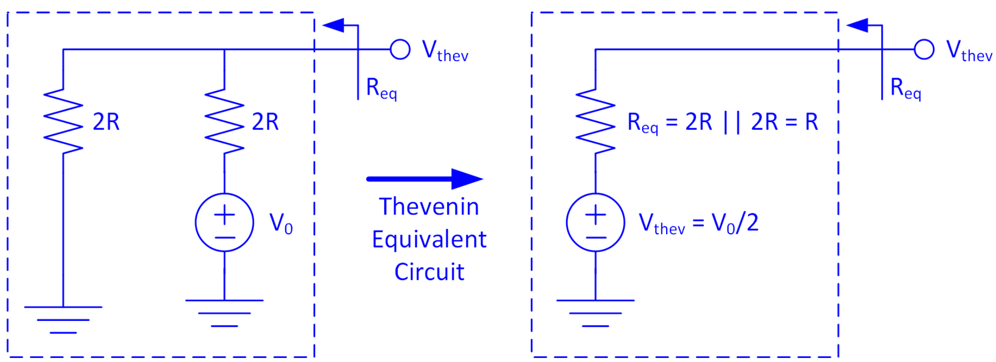


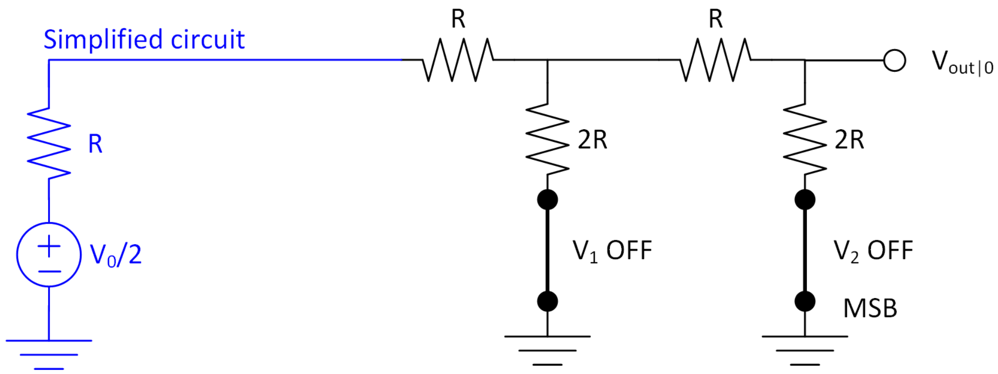
Let's first solve for $V_{\text{out}\,|\,0}$. To do so, re-draw the circuit with the independent voltage source V_0 on and all the other independent sources, voltage sources V_1 and V_2, off. Recall that when we turn a voltage source off, we treat it as a short circuit, i.e. 0 Volts (as opposed to turning a current source off, which we treat as an open circuit, i.e. 0 Amps). We show this circuit below.



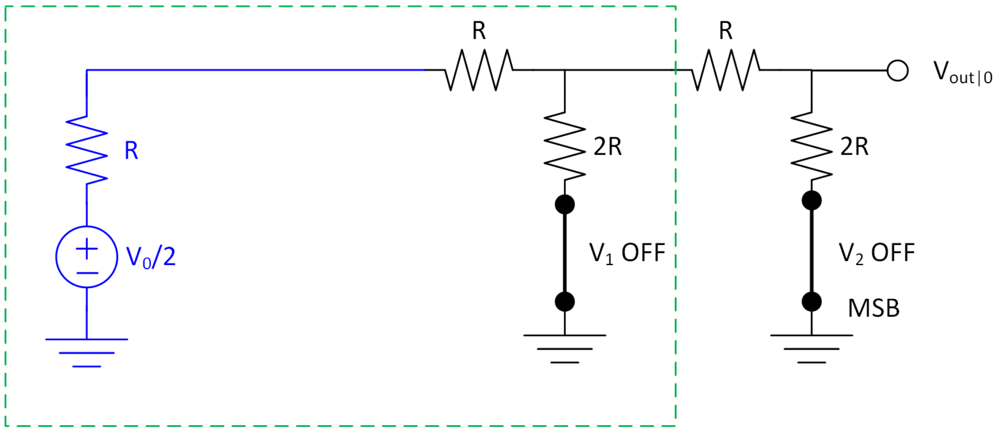
We then want to solve for $V_{\text{out}\,|\,0}$ by using Thevenin equivalent circuits to simplify the problem. We will conduct Thevenin simplification twice. First, draw a bounding box around the components we want to simplify. How you draw this box depends on how much you want to simplify at once (you get better with practice). We choose to draw the blue dotted box, find the corresponding Thevenin circuit, and replace the original components with their equivalent circuit.

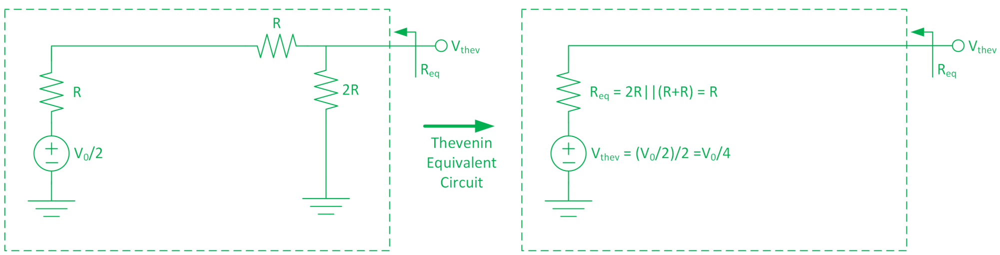


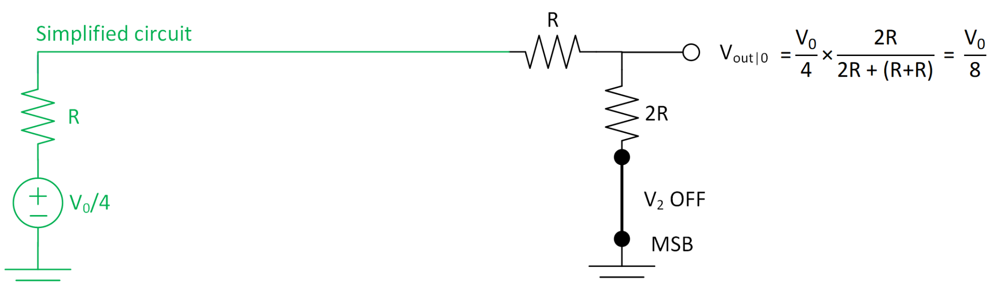




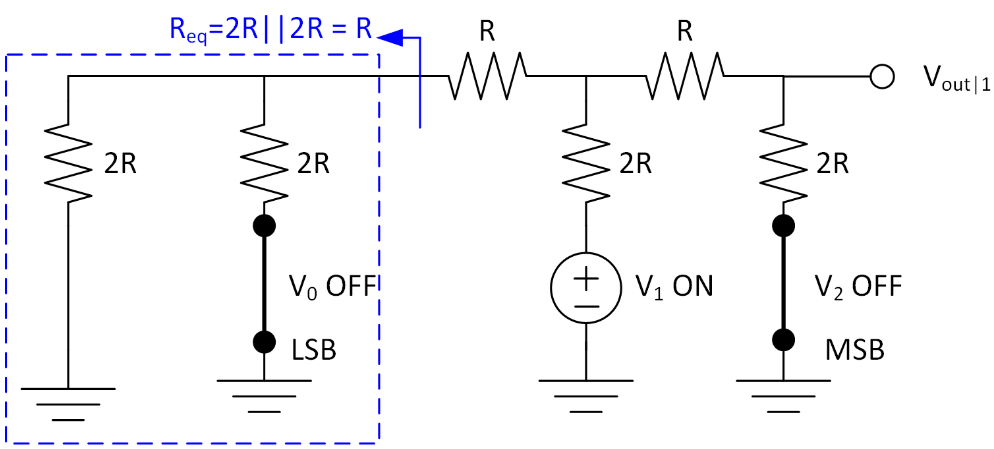
We then conduct a second round of Thevenin using the green dotted box below. From the resulting simplified circuit, we see what's left is a straightforward voltage divider, and solve for $V_{\text{out}|0}$ directly.

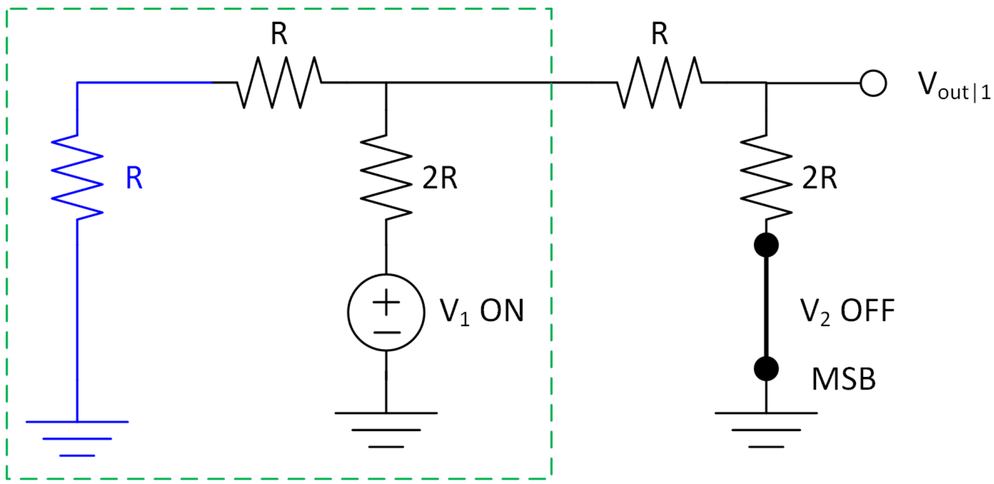


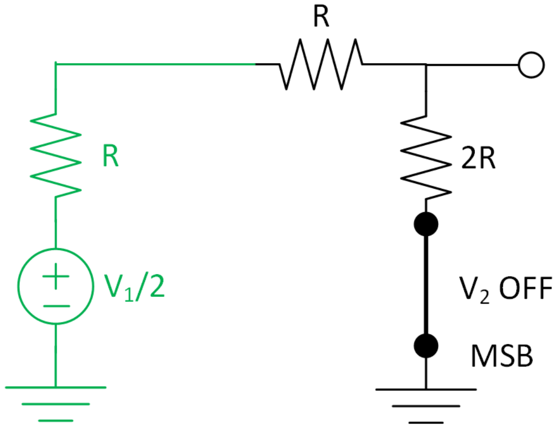




Now that we are done with the independent source V_0, we can repeat this for the other two independent sources. To find V_{out} due to independent source V_1 only, i.e. $V_{out|1}$:

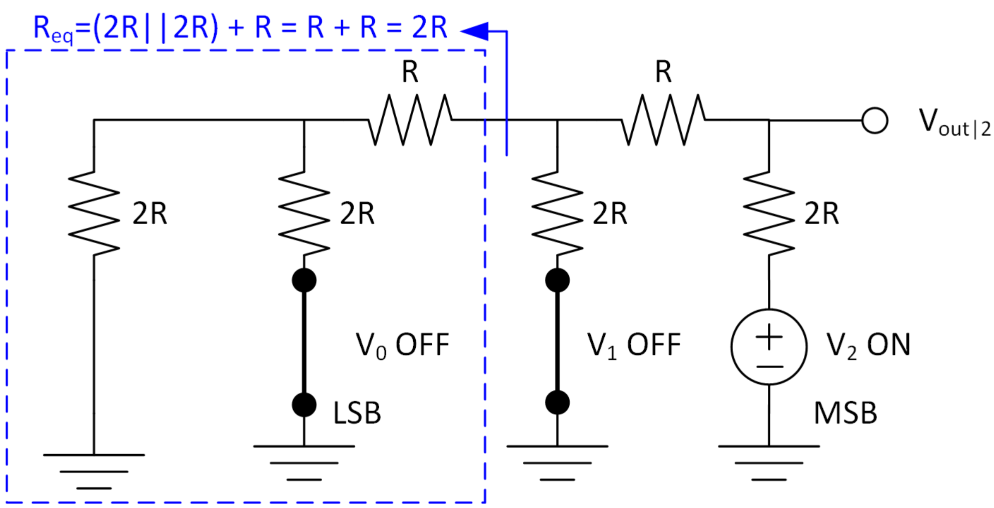


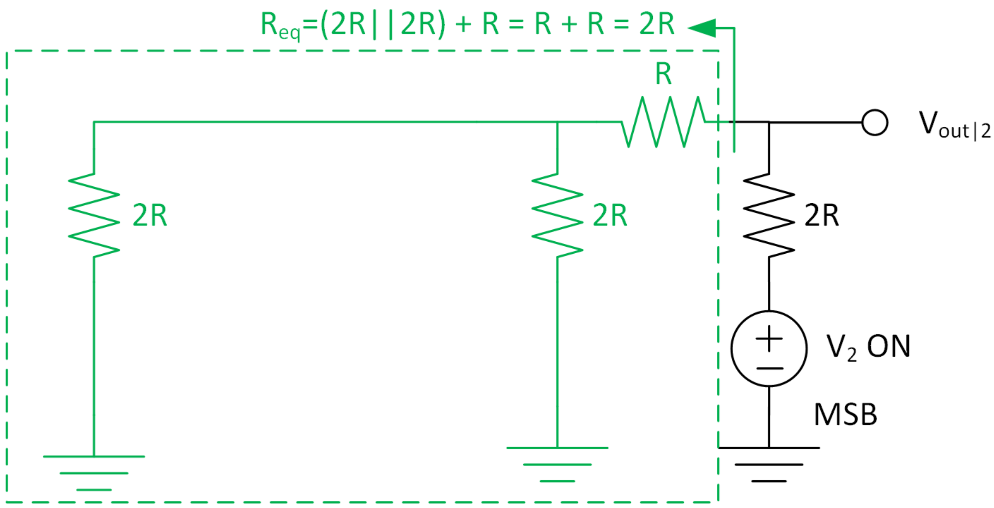


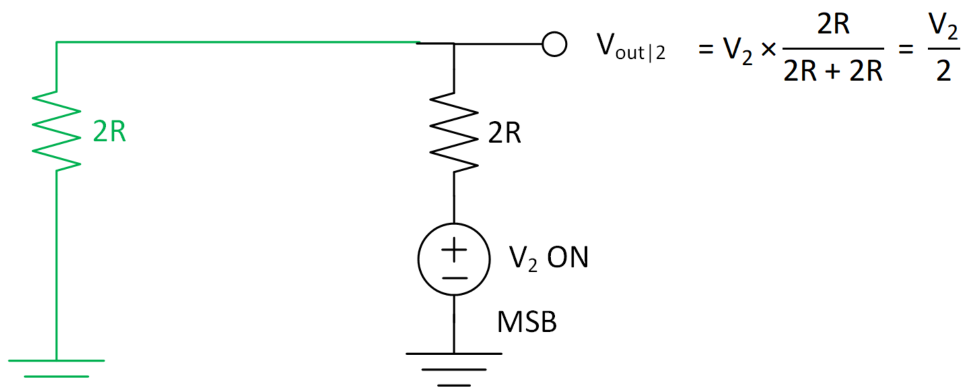


$$V_{out|1} = \frac{V_1}{2} \times \frac{2R}{2R + (R+R)} = \frac{V_1}{4}$$

Similarly, we can solve for $V_{out|2}$:







Putting it together, we have:

$$V_{\text{out}} = V_{\text{out}\,|\,0} + V_{\text{out}\,|\,1} + V_{\text{out}\,|\,2} = \frac{V_0}{8} + \frac{V_1}{4} + \frac{V_2}{2} \tag{10}$$

If we substitute for $V_0 = b_0 V_{\text{DD}}$, $V_1 = b_1 V_{\text{DD}}$, $V_2 = b_2 V_{\text{DD}}$, we get:

$$V_{\text{out}} = \frac{b_0 V_{\text{DD}}}{8} + \frac{b_1 V_{\text{DD}}}{4} + \frac{b_2 V_{\text{DD}}}{2} \tag{11}$$

which is the same result we found using KCL and Gaussian elimination in (8).

If you've made it this far, well done! Here's a challenge question for you. If instead of binary, imagine we are dealing with ternary numbers, i.e. $V_0 = t_0 V_{\text{DD}}$, $V_1 = t_1 V_{\text{DD}}$, $V_2 = t_2 V_{\text{DD}}$ where t_0, t_1, and t_2 can be -1, 0, or 1 ("ternary" means three possible values per digit). Assume we also want the output voltage levels to be evenly-spaced for all 27 $(= 3^3)$ possible 3-digit ternary numbers. Can you design a resistor DAC to achieve this?

(b) **If $b_2, b_1, b_0 = 0, 1, 1$, what is V_{out}?** Express your answer in terms of V_{DD}.

Solution: Plugging into the equation (8) from part (a), we get

$$V_{\text{out}} = \frac{3V_{\text{DD}}}{8}. \tag{12}$$

(c) **If $b_2, b_1, b_0 = 1, 0, 1$, what is V_{out}?** Express your answer in terms of V_{DD}.

Solution: Plugging into the equation (8) from part (a), we get

$$V_{\text{out}} = \frac{5V_{\text{DD}}}{8}. \tag{13}$$

(d) **If $b_2, b_1, b_0 = 1, 1, 0$, what is V_{out}?** Express your answer in terms of V_{DD}.

Solution: Plugging into the equation (8) from part (a), we get

$$V_{\text{out}} = \frac{3V_{\text{DD}}}{4}. \tag{14}$$

(e) **If $b_2, b_1, b_0 = 1, 1, 1$, what is V_{out}?** Express your answer in terms of V_{DD}.

Solution: Plugging into the equation (8) from part (a), we get

$$V_{\text{out}} = \frac{7V_{\text{DD}}}{8}. \tag{15}$$

(f) **Explain how your results above show that the resistive DAC converts the 3-bit binary number (b_2, b_1, b_0) to the output analog voltage V_{out}.**

Solution: Every increment of $\frac{1}{8}V_{\text{DD}}$ on V_{DD} represents an increment of 1 to the 3-bit binary number $(b_2 b_1 b_0)$.

Alternatively, you can view V_{DD} as being 1 and then these are the first binary digits after the "decimal point."

For example, if $V_{\text{out}} = \frac{5}{8}V_{\text{DD}}$, the input was 5 in binary (1 0 1)$\rightarrow$ $(b_2 = 1\ b_1 = 0\ b_0 = 1)$.

2. Hambley P3.16

A capacitance and the current through it are shown in Figure 1 and Figure 2 respectively. At $t = 0$, the voltage is $v_C(0) = 10\,\text{V}$. Sketch the voltage, power, and stored energy to scale versus time.

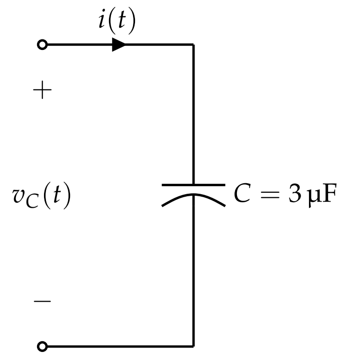


Figure 1: Circuit for P3.16

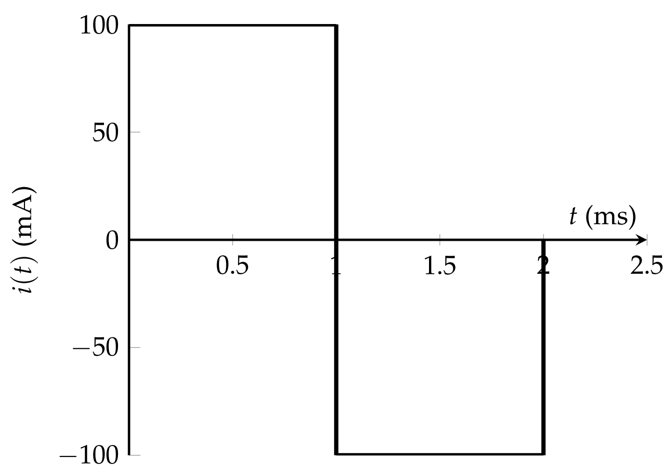


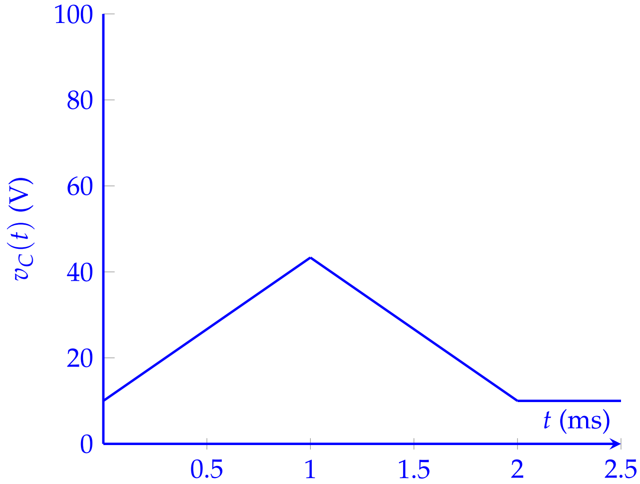
Figure 2: Current vs Time for P3.16

Solution: We can apply the capacitor differential equation and integrate both sides, yielding

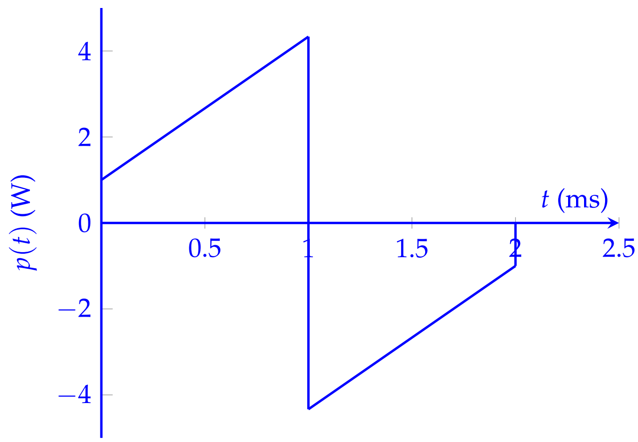
$$v_C(t) = \frac{1}{C}\int_0^t i(t')\,\mathrm{d}t' + v(0) \tag{16}$$

$$= \left(\frac{1}{3} \times 10^6\right)\int_0^t i(t')\,\mathrm{d}t' + 10 \tag{17}$$

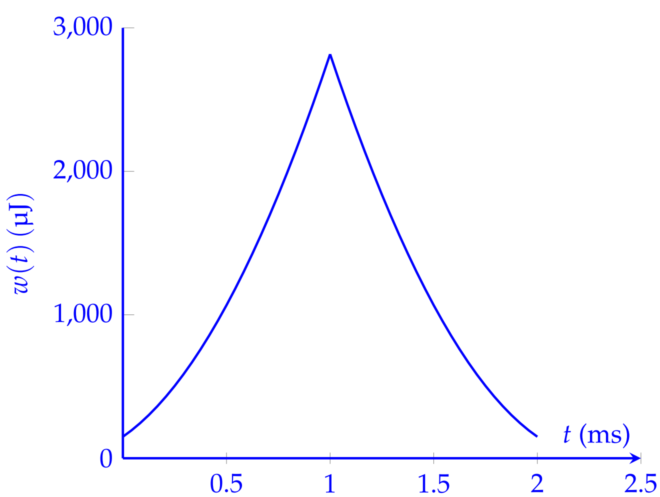
The plot of this is as follows:



Next, power is the pointwise multiplication of the above graph with Figure 2, so we have



Lastly, the stored energy is given by $\frac{1}{2}C(v_C(t))^2 = 1.5 \times 10^{-6} \times (v_C(t))^2$, so we have the following plot



3. Hambley P4.3 and P4.4

(a) The initial voltage across the capacitor shown in Figure 3 is $v_c(0+) = 0$. Find an expression for the voltage across the capacitor as a function of time, and sketch it to scale versus time.

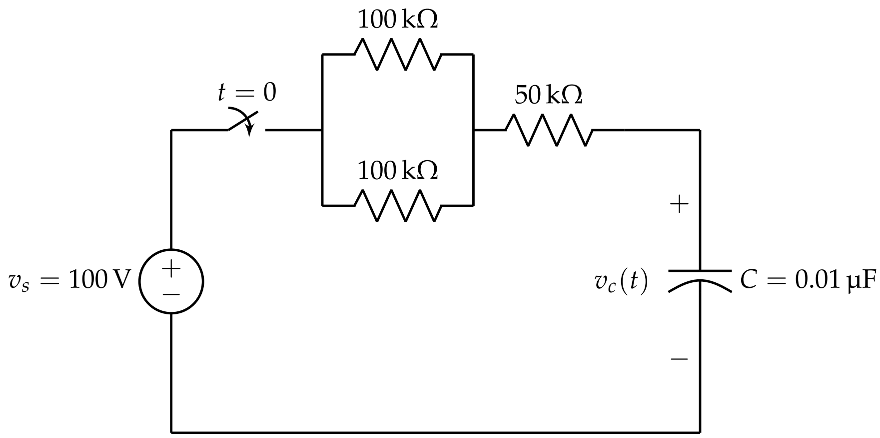
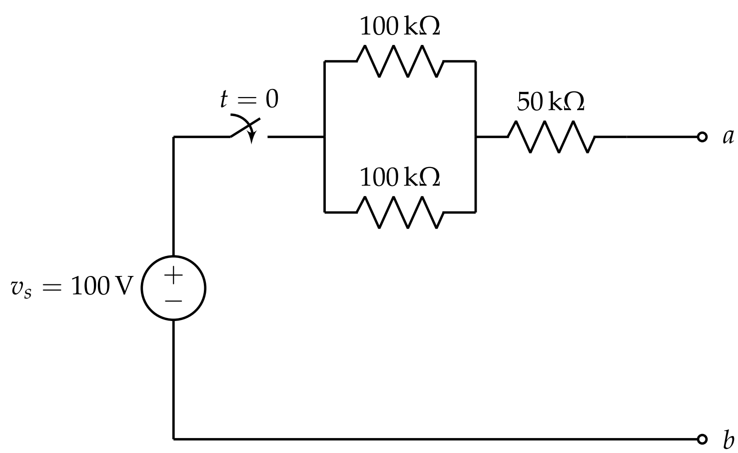


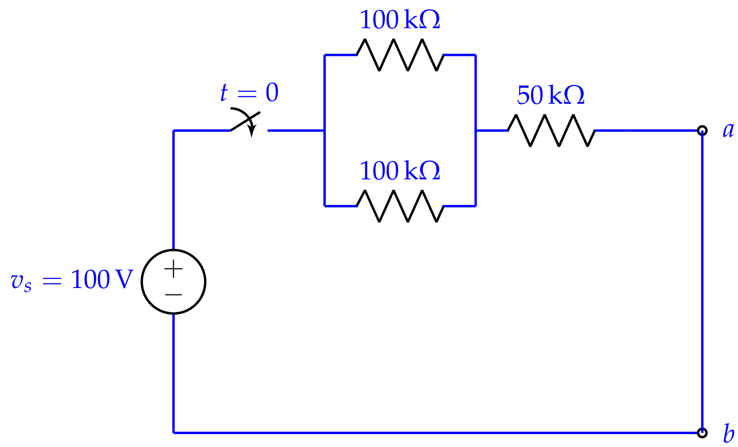
Figure 3: P4.3 Modified

(HINT: Consider simplifying the circuit using Thevenin equivalent circuits. That is, consider the following circuit, which is exactly Figure 3 without the capacitor:

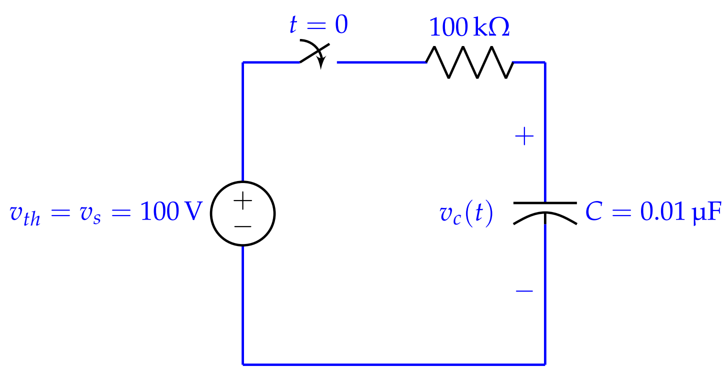


Find a Thevenin equivalent circuit and use this to simplify Figure 3.)

Solution: First, we can follow the hint to simplify the circuit. We can find V_{th} by measuring the voltage across terminals a and b. Since there is no current flowing in the circuit, $v_{th} = v_s = 100\,\text{V}$. Next, to find i_{no}, we connect terminals a and b as follows:



We can find the current between terminals a and b by first combining the resistors to find an equivalent resistance. Namely, we combine the two $100\,\mathrm{k\Omega}$ parallel resistors to obtain a single $50\,\mathrm{k\Omega}$ resistor, and we combine this resistor in series with the other $50\,\mathrm{k\Omega}$ resistor to obtain an equivalent resistance of $100\,\mathrm{k\Omega}$. Thus, $i_{no} = \frac{V_{th}}{100\,\mathrm{k\Omega}} = 1\,\mathrm{mA}$. To find R_{th}, we compute $R_{th} = \frac{v_{th}}{i_{no}} = 100\,\mathrm{k\Omega}$. Hence, an equivalent circuit to the one shown in Figure 3 is



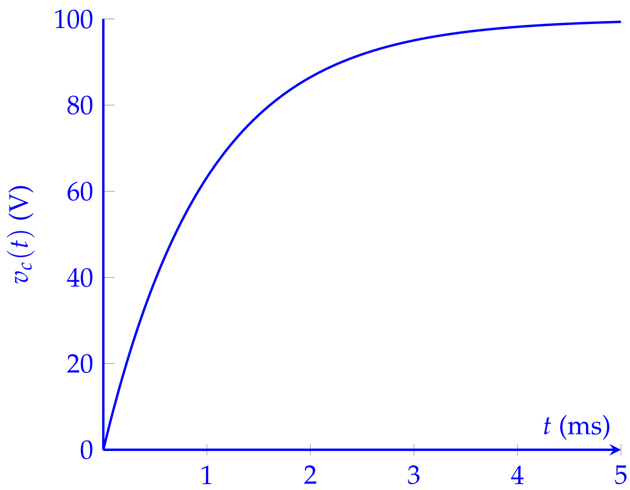
Now, we can solve for $v_c(t)$. Recall the derivation performed in lecture:

$$v_c(t) = v_s\left(1 - \mathrm{e}^{-\frac{t}{\tau}}\right) + v_c(0+)\mathrm{e}^{-\frac{t}{\tau}} \tag{18}$$

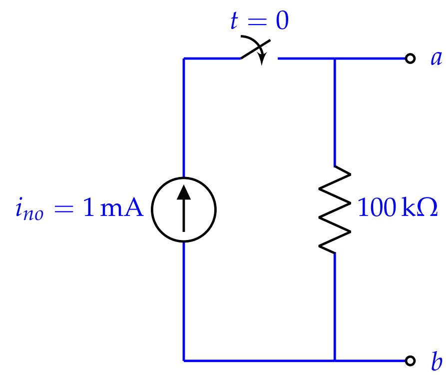
where $\tau = RC = 1\,\mathrm{ms}$. In this specific case, we have $v_c(0+) = 0$ and $v_s = 100$, so we are left with

$$v_c(t) = 100\left(1 - \mathrm{e}^{-\frac{t}{10^{-3}}}\right) \tag{19}$$

The graph of this is plotted below:



Note: You can also simplify the circuit in the hint using a current source, which would give you



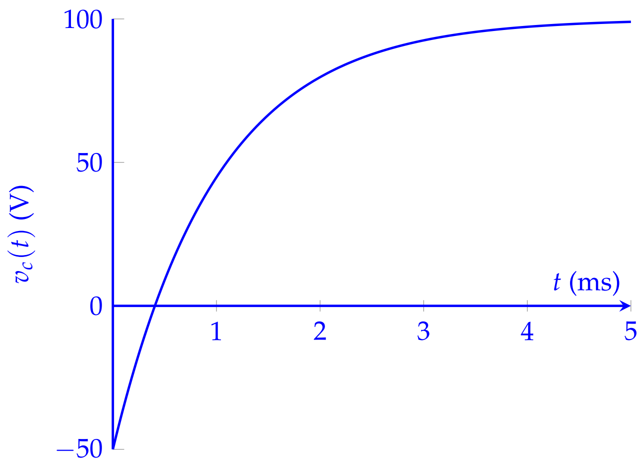
If you obtained the correct answer using this equivalent circuit, you can award yourself full credit.

(b) Repeat part (a) for an initial voltage $v_c(0+) = -50\,\text{V}$.

Solution: Using the same solution as above but setting $v_c(0+) = -50\,\text{V}$, we end up with

$$v_c(t) = 100\left(1 - e^{-\frac{t}{10^{-3}}}\right) - 50e^{-\frac{t}{10^{-3}}} \tag{20}$$

which is plotted below:


100
50
0
−50
$v_c(t)$ (V)
t (ms)
1
2
3
4
5

4. Capacitor Energy

Say a series R-C circuit is supplied by a constant voltage V. At $t = 0$, voltage across the capacitor was 0. We know how to find the expression for capacitor voltage as a function of time. Using this expression,

(a) Find the expression for total stored energy at $t = \infty$, $w_s = \int_0^\infty v(t)i(t)\,\mathrm{d}t$ (i.e., at steady state).

Solution: Applying the given formula, we have

$$\begin{aligned} w_s &= \int_0^\infty v(t)i(t)\,\mathrm{d}t && (21)\\ &= C\int_0^\infty v\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t && (22)\\ &= \frac{CV^2}{RC}\int_0^\infty \left(1-\mathrm{e}^{-\frac{t}{RC}}\right)\mathrm{e}^{-\frac{t}{RC}}\,\mathrm{d}t && (23)\\ &= \frac{CV^2}{RC}(-RC)\left(-1+\frac{1}{2}\right) && (24)\\ &= \frac{1}{2}CV^2 && (25) \end{aligned}$$

(b) We know that when a current flows through a resistor, we dissipate energy at a rate of i^2R. Using this relation, find the total dissipated energy at $t = \infty$, $w_d = \int_0^\infty i^2R\,\mathrm{d}t$ (i.e., at steady state).

Solution: First we can solve for the current, i.e.,

$$i = C\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{CV}{RC}\mathrm{e}^{-\frac{t}{RC}} \qquad (26)$$

Next, we can apply it to the given function for energy dissipated to obtain

$$\begin{aligned} w_d &= \int_0^\infty i^2R\,\mathrm{d}t && (27)\\ &= R\frac{C^2V^2}{R^2C^2}\int_0^\infty \mathrm{e}^{-\frac{2t}{RC}}\,\mathrm{d}t && (28)\\ &= R\frac{C^2V^2}{R^2C^2}\left(\frac{RC}{2}\right) && (29)\\ &= \frac{1}{2}CV^2 && (30) \end{aligned}$$

(c) Find the total energy taken from the source, noting that some of it was stored and some of it was dissipated.

Solution: Adding the results from the previous two parts, we have that the total energy from the source is CV^2.

(d) Does the result in part (b) vary if $R = 100\,\Omega$ vs $R = 1\,\mathrm{k}\Omega$? What about $R = 0\,\Omega$? *OPTIONAL: Can you explain this result?*

Solution: The total energy dissipated is independent of R. This means that even if $R = 0$, the dissipation is $\frac{1}{2}CV^2$.

Essentially, when a circuit is suddenly turned ON, the large $\frac{\mathrm{d}\xi}{\mathrm{d}t}$, where ξ is an electric field, leads to radiation through power which is lost. Understanding this more deeply requires knowledge of electromagnetcis, which is the topic of EE118. This is also at the heart of how antennas are able to radiate energy into the atmosphere.

5. Hambley P4.7

The capacitor shown in Figure 4 is charged to a voltage of 50 V prior to $t = 0$.

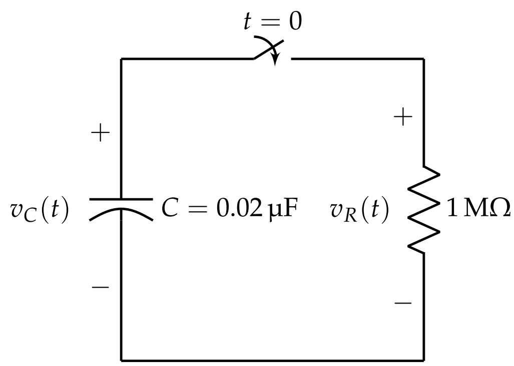


Figure 4: P4.7

(a) Find expressions for the voltage across the capacitor $v_C(t)$ and the voltage across the resistor $v_R(t)$.

Solution: We have that $RC = 0.02\,\text{s}$. Hence, using the derivation from lecture,

$$v_C(t) = \begin{cases} 50 & t < 0 \\ 50\mathrm{e}^{-\frac{t}{0.02}} = 50\mathrm{e}^{-50t} & t > 0 \end{cases} \tag{31}$$

and

$$v_R(t) = \begin{cases} 0 & t < 0 \\ 50\mathrm{e}^{-\frac{t}{0.02}} = 50\mathrm{e}^{-50t} & t > 0 \end{cases} \tag{32}$$

(b) Find an expression for the power delivered to the resistor.

Solution: We have that

$$p_R(t) = \frac{v_R(t)^2}{R} = \frac{2500}{10^6}\mathrm{e}^{-100t} = 2.5 \times 10^{-3}\mathrm{e}^{-100t}\,\text{W} \tag{33}$$

for $t > 0$, and $p_R(t) = 0\,\text{W}$ for $t < 0$

(c) Integrate the power from $t = 0$ to $t = \infty$ to find the energy delivered.

Solution: We have that

$$W = \int_0^\infty p_R(t)\,\mathrm{d}t \tag{34}$$

$$= 2.5 \times 10^{-3} \int_0^\infty \mathrm{e}^{-100t}\,\mathrm{d}t \tag{35}$$

$$= -2.5 \times 10^{-5}\left(\mathrm{e}^{-100t}|_0^\infty\right) \tag{36}$$

$$= 25\,\mu\text{J} \tag{37}$$

(d) Show that the energy delivered to the resistor is equal to the energy stored in the capacitor prior to $t = 0$.

Solution: The initial energy stored in the capacitance is

$$W = \frac{1}{2}C(v_C(0))^2 \tag{38}$$
$$= \frac{1}{2} \times 0.02 \times 10^{-6} \times 50^2 \tag{39}$$
$$= 25\,\mu\text{J} \tag{40}$$

6. Hambley P4.46

Consider the circuit shown in Figure 5. The voltage source is known as a **ramp function**, which is defined by

$$v(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases} \tag{41}$$

Assume that $v_C(0) = 0$. Derive an expression for $v_C(t)$ for $t \geq 0$. Sketch $v_C(t)$ to scale versus time.

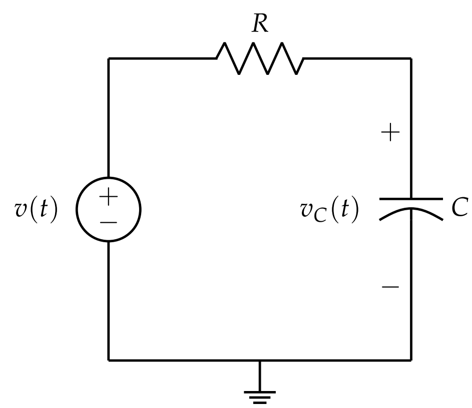


Figure 5: P4.46

Solution: From KCL, we have the following equations:

$$\frac{v(t) - v_C(t)}{R} = C\frac{\mathrm{d}v_C}{\mathrm{d}t} \tag{42}$$

$$\frac{\mathrm{d}v_C}{\mathrm{d}t} + \frac{1}{RC}v_C(t) = \frac{1}{RC}v(t) \tag{43}$$

We can define our integrating factor as $f(t) = \mathrm{e}^{\int \frac{1}{RC}\mathrm{d}t} = \mathrm{e}^{\frac{t}{RC}}$ which leaves us with

$$\mathrm{e}^{\frac{t}{RC}}\left(\frac{\mathrm{d}v_C}{\mathrm{d}t} + \frac{1}{RC}v_C(t)\right) = \mathrm{e}^{\frac{t}{RC}}\left(\frac{1}{RC}v(t)\right) \tag{44}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\frac{t}{RC}}v_C(t)\right) = \mathrm{e}^{\frac{t}{RC}}\left(\frac{1}{RC}v(t)\right) \tag{45}$$

Integrating both sides, we obtain

$$\int \frac{\mathrm{d}}{\mathrm{d}t'}\left(\mathrm{e}^{\frac{t'}{RC}}v_C(t')\right)\mathrm{d}t' = \int \mathrm{e}^{\frac{t'}{RC}}\left(\frac{1}{RC}v(t')\right)\mathrm{d}t' \tag{46}$$

$$\mathrm{e}^{\frac{t}{RC}}v_C(t) - C_1 = \int \mathrm{e}^{\frac{t'}{RC}}\left(\frac{1}{RC}v(t')\right)\mathrm{d}t' \tag{47}$$

$$v_C(t) = C_1\mathrm{e}^{-\frac{t}{RC}} + \mathrm{e}^{-\frac{t}{RC}}\int \mathrm{e}^{\frac{t'}{RC}}\left(\frac{1}{RC}v(t')\right)\mathrm{d}t' \tag{48}$$

for some arbitrary constant C_1 (that we will eventually solve for). Assuming $v(t) = t$ and writing $\tau = RC$, we obtain

$$\mathrm{e}^{-\frac{t}{\tau}}\int \mathrm{e}^{\frac{t'}{\tau}}\left(\frac{1}{\tau}v(t')\right)\mathrm{d}t' = \mathrm{e}^{-\frac{t}{\tau}}\int \mathrm{e}^{\frac{t'}{\tau}}\left(\frac{t'}{\tau}\right)\mathrm{d}t' = t - \tau + \tau\mathrm{e}^{-\frac{t}{\tau}} \tag{49}$$

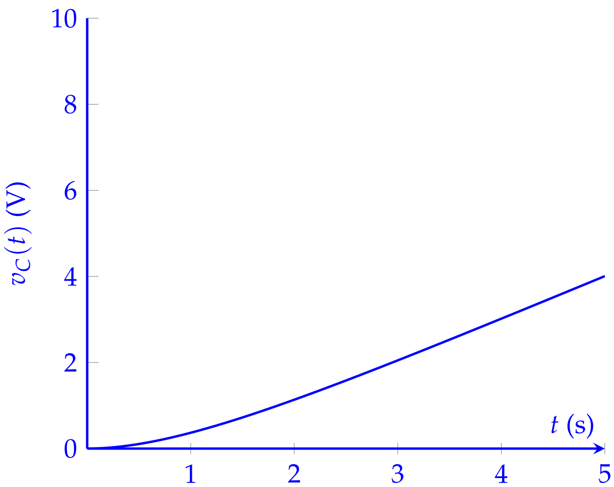
Then,

$$v_C(t) = (C_1 + \tau)e^{-\frac{t}{\tau}} + t - \tau \tag{50}$$

To find C_1, we plug in $t = 0$ to the above equation and obtain $C_1 = v_C(0) = 0$, so the final expression is

$$v_C(t) = \tau e^{-\frac{t}{\tau}} + t - \tau \tag{51}$$

A sketch of this function might look like the following (when $\tau = 1$):



Contributors:

- Edward Wang.
- Ayan Biswas.
- Wahid Rahman.
- Anish Muthali.

