Homework 3

This homework is due on Friday, February 10, 2023 at 11:59PM. Self-grades and HW Resubmissions are due the following Friday, February 17, 2023 at 11:59PM.

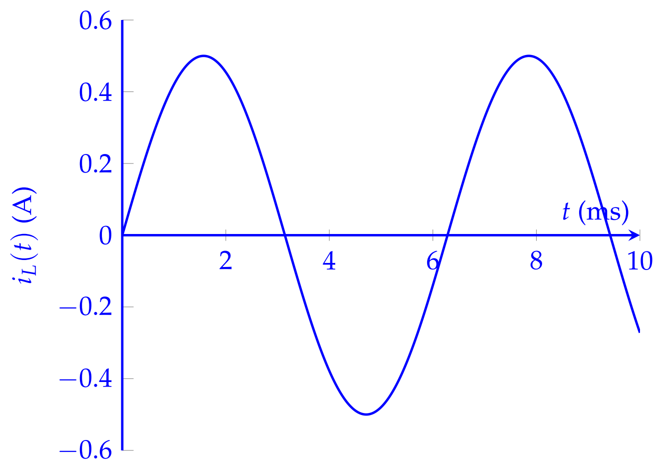
1. **(OPTIONAL) Study Group Reassignment**

   We hope your study groups from the beginning of the semester have been going well! If you did not fill out the original matching form and would now like to join a group, or if your current study group is not meeting your needs, you can request a new study group via this form. Requests for new study groups are due Friday at 11:59 PM.

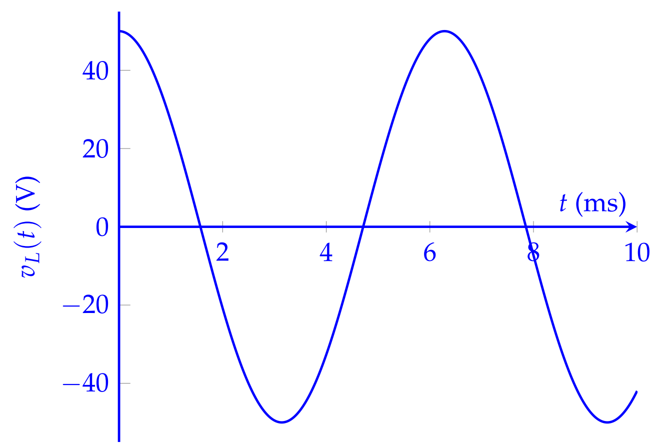
2. Hambley P3.49

The current in a 100 mH inductance is given by $0.5 \sin(1000t)$ A. Find expressions and sketch the waveforms to scale for the voltage, power, and stored energy, allowing t to range from 0 to 3π ms. The argument of the sine function is in radians.

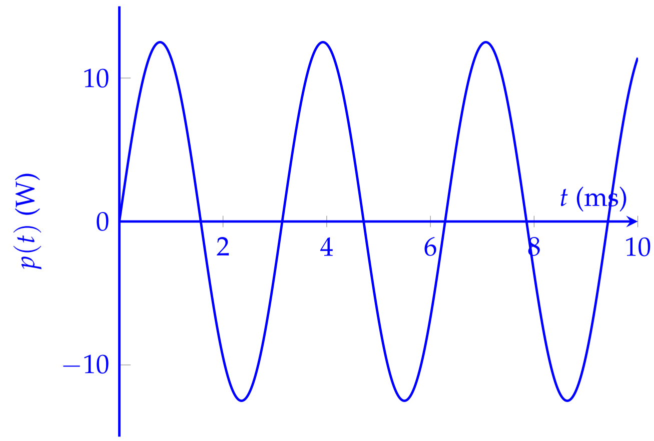
Solution: Below is the plot for $i_L(t)$:



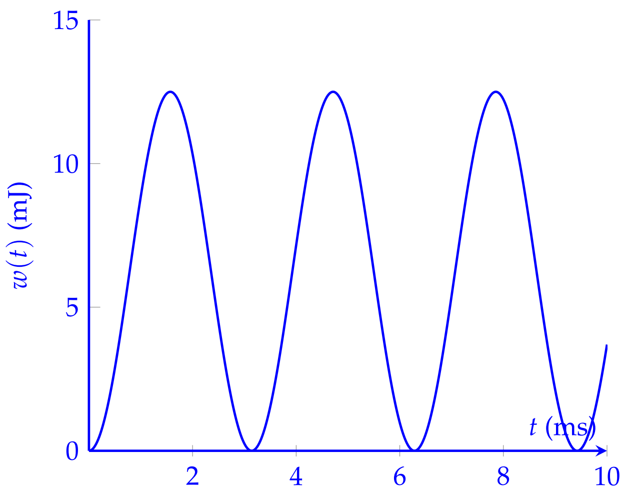
Using this and the fact that $v_L(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = 50\cos(1000t)$, we have the following plot for voltage:



Next, we have $p(t) = v_L(t)i_L(t) = 25\cos(1000t)\sin(1000t) = 12.5\sin(2000t)$, which is plotted below:



Lastly, we have $w(t) = \frac{1}{2}L(i_L(t))^2 = 0.0125\sin^2(1000t)$, which is plotted below:



3. Hambley P3.55

What value of inductance corresponds to an open circuit, assuming zero initial current? Explain your answer. Repeat for a short circuit.

Solution: In an open circuit, $i = 0$, we have that

$$i = \frac{1}{L}\int_{t_0}^{t} v(t')\,\mathrm{d}t' + i(t_0) \tag{1}$$

and $i(t_0) = 0$. Hence, it must be the case that $L \to \infty$.
In a short circuit, $v = 0$, so we have

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t} = 0 \tag{2}$$

so $L = 0$.

4. Hambley P3.74

A pair of mutually coupled inductances has $L_1 = 2\,\text{H}$, $L_2 = 1\,\text{H}$, $i_1 = 2\cos(1000t)\text{A}$, $i_2 = 0$, and $v_2 = 2000\sin(1000t)\text{V}$. (The arguments of the sine and cosine functions are in radians.) Find $v_1(t)$ and the magnitude of the mutual inductance.

Solution: In general, we have

$$v_1(t) = L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} \pm M\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{3}$$

$$v_2(t) = \pm M\frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{4}$$

Substituting the given information, we have

$$v_1(t) = -4\times 10^3\sin(1000t) \tag{5}$$

$$2000\sin(1000t) = \mp M\times 2000\sin(1000t) \tag{6}$$

We deduce that $M = 1\,\text{H}$. Furthermore, because the lower of the two algebraic signs applies, we know that the currents are referenced into unlike terminals.

5. Hambley P4.21

Solve for the steady-state values of i_1, i_2, and i_3 in the circuit shown in Figure 1.

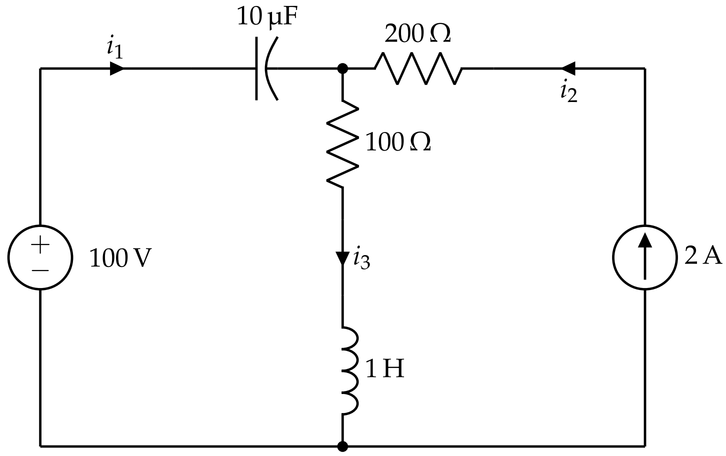
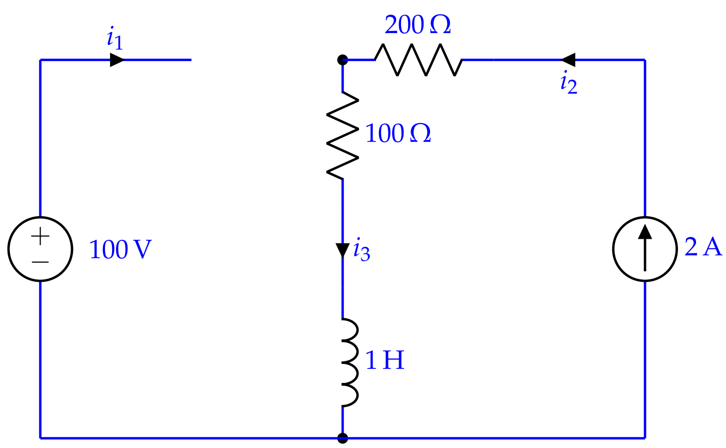


Figure 1: P4.21

Solution: At steady state, the equivalent circuit is



so $i_1 = 0$ and $i_2 = i_3 = 2\,\text{A}$.

6. Hambley P4.41

Determine expressions for and sketch $v_R(t)$ to scale versus time for the circuit of Figure 2. The circuit is operating in steady state with the switch closed prior to $t = 0$. Consider the time interval $0 \le t \le 1$ ms.

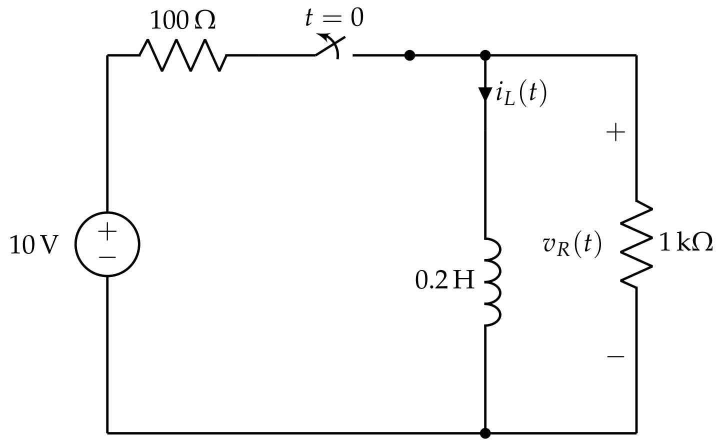
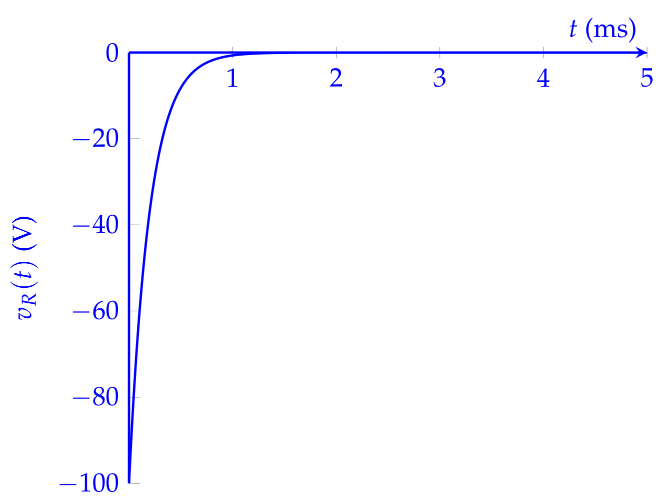


Figure 2: P4.41

Solution: In the steady state the inductor acts as a short circuit (essentially shorting the 1 kΩ resistance on the right). Then $i_L = \frac{10}{100} = 0.1$ A.

As the switch is opened, the circuit essentially becomes a R-L circuit with an initial voltage of $v_R(0) = -1000 \times 0.1 = -100$ V. Then from $L\frac{\mathrm{d}i}{\mathrm{d}t} + iR = 0$, we get $i = i(0)\mathrm{e}^{-\frac{R}{L}t}$. Thus, $v_R(t) = -100\mathrm{e}^{-5000t}$. A plot of this is shown below:



7. Hambley P4.61

A DC source is connected to a series RLC circuit by a switch that closes at $t = 0$, as shown in Figure 3. The initial conditions are $i(0+) = 0$ and $v_C(0+) = 25$. Write the differential equation for $v_C(t)$. Solve for $v_C(t)$ given that $R = 80\,\Omega$.

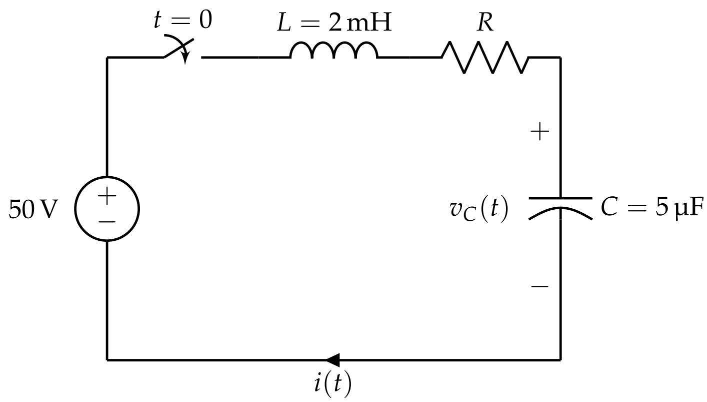


Figure 3: P4.61

Solution: We can apply KVL to the circuit to obtain:

$$50 = v_L(t) + v_R(t) + v_C(t) \tag{7}$$

$$= L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + i(t)R + v_C(t) \tag{8}$$

$$= L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + RC\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + v_C(t) \tag{9}$$

Now, we have that $L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = L\frac{\mathrm{d}}{\mathrm{d}t}\left(C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t}\right) = LC\frac{\mathrm{d}^2v_C(t)}{\mathrm{d}t^2}$. Plugging this in, we get

$$LC\frac{\mathrm{d}^2v_C(t)}{\mathrm{d}t^2} + RC\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + v_C(t) = 50 \tag{10}$$

$$\frac{\mathrm{d}^2v_C(t)}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + \frac{1}{LC}v_C(t) = \frac{50}{LC} \tag{11}$$

We have a second order differential equation, so our solution will be of the form $v_C(t) = v_{C_P}(t) + v_{C_C}(t)$, where $v_{C_P}(t)$ is the particular solution and $v_{C_C}(t)$ is the complementary solution. Here, we have a DC forcing function (i.e., $f(t) = \frac{50}{LC}$). Hence, the particular solution would be the solution if we replaced inductors with short circuits and capacitances with open circuits. This yields $v_{C_P}(t) = 50$. To find ζ and τ, we can pattern match $\tau = \sqrt{\frac{1}{LC}}$ and $\zeta\tau = \frac{R}{2L} \implies \zeta = \frac{\frac{R}{2L}}{\sqrt{\frac{1}{LC}}} = 2$ from Note 3. Since $\zeta > 1$, the complementary solution will be of the form

$$v_{C_C}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{12}$$

where $s_1 = -\zeta\tau + \tau\sqrt{\zeta^2 - 1} = -2679.49$ and $s_2 = -\zeta\tau - \tau\sqrt{\zeta^2 - 1} = -37320.5$. Hence, the final solution is of the form

$$v_C(t) = v_{C_P}(t) + v_{C_C}(t) = 50 + K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{13}$$

To find K_1 and K_2, we can utilize the fact that $v_C(0) = 0$ and $\frac{\mathrm{d}v_C(t)}{\mathrm{d}t}|_{t=0} = \frac{i(0)}{C} = 0$. Plugging these in, we get the following system:

$$v_C(0) = 25 = 50 + K_1 + K_2 \tag{14}$$

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t}|_{t=0} = 0 = s_1 K_1 + s_2 K_2 \tag{15}$$

Solving this system of equation yields $K_1 = -26.93$ and $K_2 = 1.93$. Hence, the final answer is

$$v_C(t) = 50 + (-26.93)\mathrm{e}^{-2679.49t} + (1.93)\mathrm{e}^{-37320.5t} \tag{16}$$

8. Hambley P4.64

Consider the circuit shown in Figure 4, with $R = 25\,\Omega$.

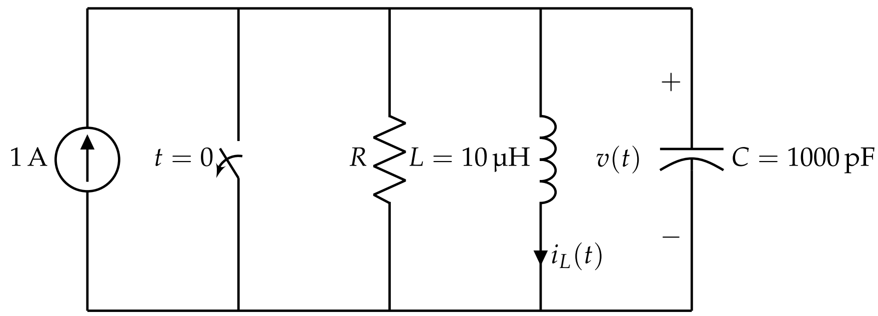


Figure 4: P4.64

(a) Compute the undamped resonant frequency, τ, and ζ.

Solution: From KCL, we have

$$1 = i_R(t) + i_L(t) + i_C(t) \tag{17}$$

$$= \frac{v(t)}{R} + i_L(t) + C\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{18}$$

Taking derivatives on both sides, we get

$$C\frac{\mathrm{d}^2v(t)}{\mathrm{d}t^2} + \frac{1}{R}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = 0 \tag{19}$$

We also know that $v(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t}$. Plugging this in, we get

$$C\frac{\mathrm{d}^2v(t)}{\mathrm{d}t^2} + \frac{1}{R}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{L}v(t) = 0 \tag{20}$$

$$\frac{\mathrm{d}^2v(t)}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{LC}v(t) = 0 \tag{21}$$

Pattern matching to Note 3, we get $\tau = \sqrt{\frac{1}{LC}} = 1 \times 10^7$ and $\zeta\tau = \frac{1}{2RC} = 2 \times 10^7 \implies \zeta = 2$. Hence, it is an overdamped circuit ($\zeta > 1$).

(b) The initial conditions are $v(0+) = 0$ and $i_L(0+) = 0$. Show that this requires $v'(0+) = 10^9\,\frac{\mathrm{V}}{\mathrm{s}}$.

Solution: We still must satisfy KCL at $t = 0+$, so we have

$$1 = i_R(0+) + i_L(0+) + i_C(0+) \tag{22}$$

$$= \frac{v(0+)}{R} + i_L(0+) + Cv'(0+) \tag{23}$$

$$= Cv'(0+) \tag{24}$$

This leaves us with $v'(0+) = \frac{1}{C} = 10^9\,\frac{\mathrm{V}}{\mathrm{s}}$.

(c) Find the particular solution for $v(t)$.

Solution: To find the particular solution, we first notice that the forcing function is $f(t) = 0$ which is a constant. Hence, we can replace capacitors with open circuits and inductors with short circuits. If we were to do this, all of the current flows through the branch with the inductor and the particular solution is $v_P(t) = 0$.

(d) Find the general solution for $v(t)$, including the numerical values of all parameters.

Solution: Since $\zeta > 1$, the complementary solution will be of the form

$$v_C(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{25}$$

where $s_1 = -\zeta\tau + \tau\sqrt{\zeta^2 - 1} = -2.68 \times 10^6$ and $s_2 = -\zeta\tau - \tau\sqrt{\zeta^2 - 1} = -3.73 \times 10^7$. Since the particular solution $v_P(t) = 0$, we have that

$$v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{26}$$

Now, we will use our initial conditions of $v(0) = 0$ and $v'(0) = 10^9$. Plugging these in, we get the following system of equations:

$$v(0) = 0 = K_1 + K_2 \tag{27}$$

$$v'(0) = 10^9 = s_1 K_1 + s_2 K_2 \tag{28}$$

Solving the system of equations yields $K_1 = 28.89$ and $K_2 = -28.89$. Thus, the final answer is

$$v(t) = 28.89 e^{(-2.68\times 10^6)t} - 28.89 e^{(-3.73\times 10^7)t} \tag{29}$$