Homework 4

This homework is due on Friday, February 17, 2023 at 11:59PM. Self-grades and HW Resubmissions are due the following Friday, February 24, 2023 at 11:59PM.

1. Tracking Terry

Terry is a mischievous child, and his mother is interested in tracking him.

(a) Terry texts his current location as a vector $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, but there is a problem! These coordinates are *not* in the standard basis, but rather in the basis $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. That is to say that the first number 2 above is how many multiples of $\vec{v}_1$ to use and the second number 3 is how many multiples of $\vec{v}_2$ to use in computing his actual location. Here, both $\vec{v}_1$ and $\vec{v}_2$ are vectors in the standard basis.

Let Terry's location in the standard basis be $\vec{x}$. Write $\vec{x}$ in terms of $\vec{v}_1$ and $\vec{v}_2$.

Solution: By definition, the first coordinate in the V basis is the coefficient of $\vec{v}_1$ and second coordinate in the V basis is the coefficient for $\vec{v}_2$. Hence

$$\vec{x} = V\vec{x}_v = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\vec{v}_1 + 3\vec{v}_2. \tag{1}$$

(b) Terry's friend tells you that Terry's location in the standard basis is $\vec{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Using this along with the previous info that Terry's location in the V basis is $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, **is it possible to determine the basis vectors $\vec{v}_1, \vec{v}_2$ Terry is using. If it is impossible to do so, explain why.**

(HINT: How many unknowns do you have? How many equations?)

Solution: Solving for the basis vectors Terry is using (or in other words the axes in his coordinate space) is the same as solving for V in the change of basis equation:

$$V\vec{x}_v = \vec{x} \tag{2}$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \tag{3}$$

There are four unknowns and only two equations, so this task is impossible.

(c) Terry's basis vectors $\vec{v}_1, \vec{v}_2$ get leaked to his mom on accident, so she knows they are

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \tag{4}$$

To hide his location, Terry wants to switch to a new coordinate system with the basis vectors

$$\vec{p}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5}$$

In order to do this, he needs a way to convert coordinates from the V basis to the P basis. Thus, **find the matrix T such that if $\vec{x}_v$ is a location expressed in V coordinates and $\vec{x}_p$ is the same location expressed in P coordinates, then $\vec{x}_p = T\vec{x}_v$.**

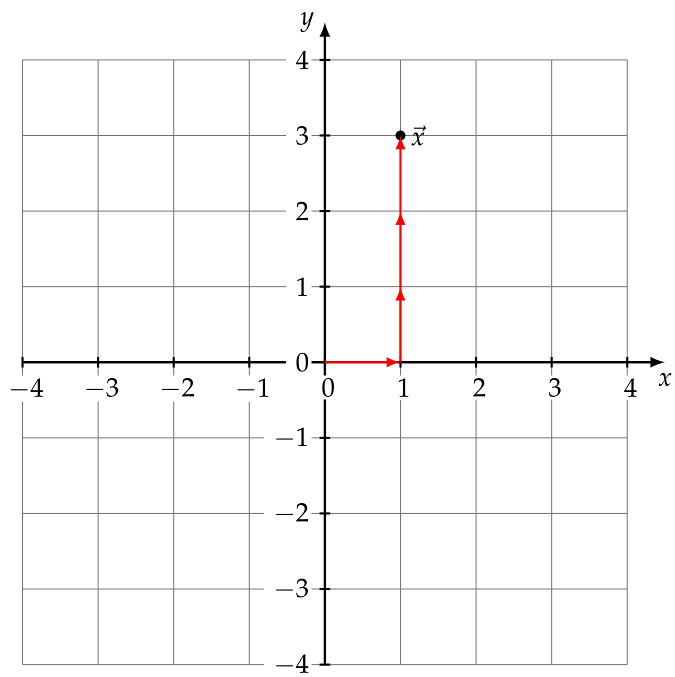
Solution: The problem can be formulated as a change of basis problem. Since both $\vec{x}_v$ and $\vec{x}_p$ correspond to the same point, converting them to the standard basis gives us

$$V\vec{x}_v = P\vec{x}_p \tag{6}$$

Since we want to find T such that $\vec{x}_p = T\vec{x}_v$, we have:

$$\begin{aligned}
\vec{x}_p &= P^{-1}V\vec{x}_v && (7)\\
T &= P^{-1}V && (8)\\
&= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} && (9)\\
&= \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} && (10)\\
&= \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} && (11)
\end{aligned}$$

(d) Terry now wants to make a map and route to where he currently is, $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. **For both the P and V bases from part 1.c, illustrate the sum of scaled basis vectors that are necessary to go from the origin to $\vec{x}$.** An example is shown below when using the standard basis. This illustrates that the same location can be represented by many different coordinate systems/bases.



Solution:

Since we know $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, and $\vec{x} = V\vec{x}_v$, we can derive:

$$\vec{x}_v = V^{-1}\vec{x} \tag{12}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{13}$$

$$= \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{14}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{15}$$

Similarly, in order to compute $\vec{x}_p$, we have:

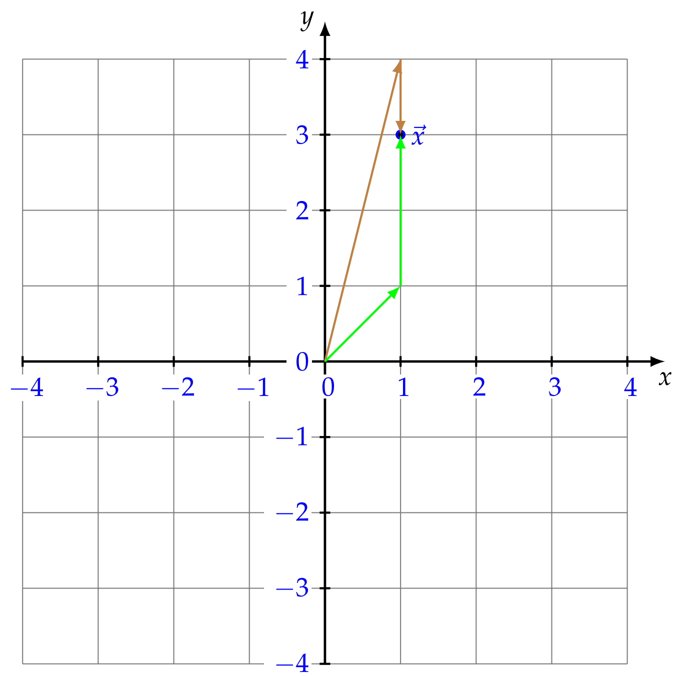
$$\vec{x}_p = P^{-1}\vec{x} \tag{16}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{17}$$

$$= \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{19}$$

Therefore, we can illustrate the sum of scaled basis vectors according to $\vec{x}_v$ (green path), and $\vec{x}_p$ (brown path).



2. Eigenvectors and Diagonalization

(a) Let A be an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define V to be a matrix with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ as its columns, $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$.
Show that $AV = V\Lambda$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues of A as its diagonal entries.

Solution:

$$\begin{aligned} AV &= A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} & (20) \\ &= \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots, A\vec{v}_n \end{bmatrix} & (21) \\ &= \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} & (22) \\ &= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} & (23) \\ &= V\Lambda & (24) \end{aligned}$$

(b) **Argue that V is invertible, and therefore**

$$A = V\Lambda V^{-1}. \tag{25}$$

(HINT: What condition on a matrix's columns means that it would be invertible? It is fine to cite the appropriate result from 16A.)

Solution: Columns of V are eigenvectors of A which are known to be linearly independent. Since V has linearly independent columns, it has full column rank, and therefore, is invertible.

$$\begin{aligned} AV &= V\Lambda & (26) \\ AVV^{-1} &= V\Lambda V^{-1} & (27) \\ A &= V\Lambda V^{-1} & (28) \end{aligned}$$

(c) **Write Λ in terms of the matrices A, V, and V^{-1}.**

Solution: We take $A = V\Lambda V^{-1}$ and apply invertible operations to both sides of the equality:

$$\begin{aligned} A &= V\Lambda V^{-1} & (29) \\ V^{-1}A &= V^{-1}V\Lambda V^{-1} & (30) \\ V^{-1}AV &= V^{-1}V\Lambda V^{-1}V & (31) \\ V^{-1}AV &= I\Lambda I & (32) \\ V^{-1}AV &= \Lambda. & (33) \end{aligned}$$

(d) A matrix A is deemed diagonalizable if there exists a square matrix U so that A can be written in the form $A = UDU^{-1}$ for the choice of an appropriate diagonal matrix D.

Show that the columns of U must be eigenvectors of the matrix A, and that the entries of D must be eigenvalues of A.

(HINT: Recall the definition of an eigenvector (i.e., $A\vec{v} = \lambda\vec{v}$). Then, recall what $U^{-1}U$ is. Lastly, consider how matrix multiplication works column-wise.)

Solution: We start with a calculation which is essentially the reverse of the calculation in part (b):

$$A = UDU^{-1} \tag{34}$$
$$AU = UDU^{-1}U \tag{35}$$
$$AU = UD. \tag{36}$$

Now let's expand the definitions of U as a square matrix and D as a diagonal matrix:

$$AU = A\begin{bmatrix}\vec{u}_1 & \dots & \vec{u}_n\end{bmatrix} \tag{37}$$
$$= \begin{bmatrix}A\vec{u}_1 & \dots & A\vec{u}_n\end{bmatrix} \tag{38}$$
$$UD = \begin{bmatrix}\vec{u}_1 & \dots & \vec{u}_n\end{bmatrix}\begin{bmatrix}d_1 & & \\ & \ddots & \\ & & d_n\end{bmatrix} \tag{39}$$
$$= \begin{bmatrix}d_1\vec{u}_1 & \dots & d_n\vec{u}_n\end{bmatrix}. \tag{40}$$

Comparing columns, we see that $A\vec{u}_i = d_i\vec{u}_i$. This is exactly the eigenvector-eigenvalue equation! In particular, this says that $\vec{u}_i$ is an eigenvector of A, with eigenvalue d_i.

The previous part shows that the *only* way to diagonalize A is using its eigenvalues/eigenvectors. Now we will explore a payoff for diagonalizing A – an operation that diagonalization makes *much* simpler.

(e) For a matrix A and a positive integer k, we define the exponent to be

$$A^k = \underbrace{A \cdot A \cdot \dots \cdot A \cdot A}_{k \text{ times}} \tag{41}$$

Let's assume that matrix A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ (i.e. the n eigenvectors are all linearly independent).

Show that A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$ and eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Conclude that A^k is diagonalizable.

Solution: Consider the i^{th} eigenvector of A, $\vec{v}_i$ and the corresponding eigenvalue λ_i.

$$A^k\vec{v}_i = A^{k-1} \cdot A\vec{v}_i \tag{42}$$
$$= A^{k-1}\lambda_i\vec{v}_i \tag{43}$$
$$= \lambda_i A^{k-2} \cdot A\vec{v}_i \tag{44}$$
$$= \lambda_i^2 A^{k-3} \cdot A\vec{v}_i \tag{45}$$

$$\vdots \tag{46}$$

$$= \lambda_i^k \vec{v}_i \tag{47}$$

Thus by definition, v_i is an eigenvector of A^k with corresponding eigenvalue λ_i^k.

Alternate solution: Since A is diagonalizable, we can express A as

$$A = V\Lambda V^{-1} \tag{48}$$

Substituting A as shown in Equation 48 in 41, we get

$$\begin{aligned}
A^k &= \underbrace{A \cdot A \cdot \dots \cdot A \cdot A}_{k \text{ times}} && (49)\\
&= \underbrace{V\Lambda V^{-1} \cdot V\Lambda V^{-1} \cdot \dots \cdot V\Lambda V^{-1} \cdot V\Lambda V^{-1}}_{k \text{ times}} && (50)\\
&= \underbrace{V\Lambda \left(V^{-1} \cdot V\right) \Lambda V^{-1} \cdot \dots \cdot V\Lambda \left(V^{-1} \cdot V\right) \Lambda V^{-1}}_{k \text{ times}} && (51)\\
&= V \underbrace{\Lambda \cdot \Lambda \cdot \dots \cdot \Lambda \cdot \Lambda}_{k \text{ times}} V^{-1} && (52)\\
&= V\Lambda^k V^{-1} && (53)
\end{aligned}$$

Since Λ is a diagonal matrix,

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix} \tag{54}$$

Thus, A^k is clearly diagonalizable, where the eigenvectors of A^k are just the eigenvectors of A, and the eigenvalues of A^k are $\lambda_1^k, \dots, \lambda_n^k$.

3. Simple Vector Differential Equations Driven by an Input

In this question, we will show the existence and uniqueness of solutions to systems of differential equations with inputs. In particular, we previously considered the scalar differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu(t) \tag{55}$$

$$x(0) = x_0 \tag{56}$$

where $u\colon \mathbb{R} \to \mathbb{R}$ is a known function of time, and we showed that there exists a unique solution to this differential equation, namely

$$x(t) := \mathrm{e}^{\lambda t}x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau \tag{57}$$

(a) Extend the solution in eq. (57) to the diagonal, vector differential equation case. **Namely, show that the solution for**

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \Lambda\vec{x}(t) + \vec{b}u(t) \tag{58}$$

is given by

$$\vec{x}(t) = \mathrm{e}^{\Lambda t}\vec{x}(0) + \int_0^t \mathrm{e}^{\Lambda(t-\tau)}\vec{b}u(\tau)\,\mathrm{d}\tau \tag{59}$$

where $\vec{x}(t) : \mathbb{R} \to \mathbb{R}^n$, $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix, $\vec{b} \in \mathbb{R}^n$, and $u(t) : \mathbb{R} \to \mathbb{R}$. For notational convenience, we define $\mathrm{e}^{\Lambda x} = \begin{bmatrix} \mathrm{e}^{\lambda_1 x} & & & \\ & \mathrm{e}^{\lambda_2 x} & & \\ & & \ddots & \\ & & & \mathrm{e}^{\lambda_n x} \end{bmatrix}$ where $\Lambda := \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$.

(HINT: You may use the fact that, if $\vec{z}(t) : \mathbb{R} \to \mathbb{R}^n$ and $\vec{z}(t) := \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix}$, it is the case that $\int_a^b \vec{z}(t)\,\mathrm{d}t = \begin{bmatrix} \int_a^b z_1(t)\,\mathrm{d}t \\ \int_a^b z_2(t)\,\mathrm{d}t \\ \vdots \\ \int_a^b z_n(t)\,\mathrm{d}t \end{bmatrix}$.)

(HINT: Consider breaking down the diagonal vector differential equation case into n different scalar cases. How can you combine the results from eq. (57) with this?)

Solution: The kth row of this differential equation will be

$$\frac{\mathrm{d}}{\mathrm{d}t}x_k(t) = \lambda_k x_k(t) + b_k u(t) \tag{60}$$

which has the solution

$$x_k(t) = \mathrm{e}^{\lambda_k t}x_k(0) + \int_0^t \mathrm{e}^{\lambda_k(t-\tau)}b_k u(\tau)\,\mathrm{d}\tau \tag{61}$$

This yields the vector solution as

$$\vec{x}(t) = \begin{bmatrix} \mathrm{e}^{\lambda_1 t}x_1(0) \\ \mathrm{e}^{\lambda_2 t}x_2(0) \\ \vdots \\ \mathrm{e}^{\lambda_n t}x_n(0) \end{bmatrix} + \begin{bmatrix} \int_0^t \mathrm{e}^{\lambda_1(t-\tau)}b_1 u(\tau)\,\mathrm{d}\tau \\ \int_0^t \mathrm{e}^{\lambda_2(t-\tau)}b_2 u(\tau)\,\mathrm{d}\tau \\ \vdots \\ \int_0^t \mathrm{e}^{\lambda_n(t-\tau)}b_n u(\tau)\,\mathrm{d}\tau \end{bmatrix} \tag{62}$$

We can simplify the first term to be

$$\begin{bmatrix} e^{\lambda_1 t} x_1(0) \\ e^{\lambda_2 t} x_2(0) \\ \vdots \\ e^{\lambda_n t} x_n(0) \end{bmatrix} = e^{\Lambda t} \vec{x}(0) \tag{63}$$

and, using the hint, we can simplify the second term as follows:

$$\begin{aligned} \begin{bmatrix} \int_0^t e^{\lambda_1 (t-\tau)} b_1 u(\tau)\, d\tau \\ \int_0^t e^{\lambda_2 (t-\tau)} b_2 u(\tau)\, d\tau \\ \vdots \\ \int_0^t e^{\lambda_n (t-\tau)} b_n u(\tau)\, d\tau \end{bmatrix} &= \int_0^t \begin{bmatrix} e^{\lambda_1 (t-\tau)} b_1 u(\tau) \\ e^{\lambda_2 (t-\tau)} b_2 u(\tau) \\ \vdots \\ e^{\lambda_n (t-\tau)} b_n u(\tau) \end{bmatrix} d\tau & (64) \\ &= \int_0^t \begin{bmatrix} e^{\lambda_1 (t-\tau)} b_1 \\ e^{\lambda_2 (t-\tau)} b_2 \\ \vdots \\ e^{\lambda_n (t-\tau)} b_n \end{bmatrix} u(\tau)\, d\tau & (65) \\ &= \int_0^t e^{\Lambda (t-\tau)} \vec{b} u(\tau)\, d\tau & (66) \end{aligned}$$

Altogether, this yields,

$$\vec{x}(t) = e^{\Lambda t} \vec{x}(0) + \int_0^t e^{\Lambda (t-\tau)} \vec{b} u(\tau)\, d\tau \tag{67}$$

(b) **Extend the result from the previous part for an arbitrary vector differenial equation given by**

$$\frac{d}{dt} \vec{x}(t) = A\vec{x}(t) + \vec{b} u(t) \tag{68}$$

where A is a *diagonalizable* matrix (but not necessarily a *diagonal* matrix). You may assume that A can be diagonalized as $A = V\Lambda V^{-1}$. For notational convenience, you may want to define $\widetilde{\vec{b}} = V^{-1}\vec{b}$.

Solution: Define $\vec{x}_\lambda(t) = V^{-1}\vec{x}(t)$. We know that the differential equation governing $\vec{x}_\lambda(t)$ will be

$$\begin{aligned} \frac{d}{dt} \vec{x}_\lambda(t) &= \Lambda \vec{x}_\lambda(t) + V^{-1}\vec{b} u(t) & (69) \\ &= \Lambda \vec{x}_\lambda(t) + \widetilde{\vec{b}} u(t) & (70) \end{aligned}$$

The solution to this, using the previous part, is

$$\vec{x}_\lambda(t) = e^{\Lambda t} \vec{x}_\lambda(0) + \int_0^t e^{\Lambda (t-\tau)} \widetilde{\vec{b}} u(\tau)\, d\tau \tag{71}$$

Hence, the solution for $\vec{x}(t)$ will be

$$\begin{aligned} \vec{x}(t) &= V e^{\Lambda t} \vec{x}_\lambda(0) + V \int_0^t e^{\Lambda (t-\tau)} \widetilde{\vec{b}} u(\tau)\, d\tau & (72) \\ &= V e^{\Lambda t} V^{-1} \vec{x}(0) + V \int_0^t e^{\Lambda (t-\tau)} \widetilde{\vec{b}} u(\tau)\, d\tau & (73) \end{aligned}$$

where we substituted $\vec{x}_\lambda(0) = V^{-1}\vec{x}(0)$.

4. Vector Differential Equations

Note: it's recommended to finish the previous question (Eigenvectors and Diagonalization) before this problem.

Consider a system of ordinary differential equations that can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) := \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\vec{x}(t) \tag{74}$$

where $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ are scalar functions of time t, and $A \in \mathbb{R}^{2\times 2}$ is a 2×2 matrix with constant coefficients. We call eq. (74) a vector differential equation.

(a) It turns out that we can actually turn all higher-order differential equations with constant coefficients into vector differential equations of the style of eq. (74).

Consider a second-order ordinary differential equation

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + a\frac{\mathrm{d}y(t)}{\mathrm{d}t} + by(t) = 0, \tag{75}$$

where $a, b \in \mathbb{R}$.

Write this differential equation in the form of (eq. (74)), by choosing appropriate variables $x_1(t)$ and $x_2(t)$.

(HINT: Your original unknown function $y(t)$ has to be one of those variables. The heart of the question is to figure out what additional variable can you use so that you can express eq. (75) without having to take a second derivative, and instead just taking the first derivative of something.)

Solution: If we set $x_1(t) = y(t)$, $x_2(t) = \frac{\mathrm{d}y(t)}{\mathrm{d}t}$, then we have

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2(t) \tag{76}$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = -a\frac{\mathrm{d}y(t)}{\mathrm{d}t} - by(t) = -ax_2(t) - bx_1(t) \tag{77}$$

We can write this in the form of eq. (74) as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{78}$$

(b) It turns out that all two-dimensional vector linear differential equations with distinct eigenvalues have a solution in the general form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_1 t} + c_1 e^{\lambda_2 t} \\ c_2 e^{\lambda_1 t} + c_3 e^{\lambda_2 t} \end{bmatrix} \tag{79}$$

where c_0, c_1, c_2, c_3 are constants, and λ_1, λ_2 are the eigenvalues of A (this can be proven by just repeating the same steps in the previous parts and using the fact that distinct eigenvalues implies linearly independent eigenvectors). Thus, an alternate way of solving this type of differential equation in the future is to now use your knowledge that the solution is of this form and just solve for the constants c_i.

Now let $a = -1$ and $b = -2$ in eq. (75), i.e.

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}y(t)}{\mathrm{d}t} - 2y(t) = 0, \tag{80}$$

Solve eq. (80) with the initial conditions $y(0) = 1$, $\frac{dy}{dt}(0) = 1$, using the general form in eq. (79). *(HINT: You get two equations using the initial conditions above. How many unknowns are here?) (HINT: Given your specific choice of x_1 and x_2 in part (a), how many unknowns are there really?)*

Solution: We have

$$\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \tag{81}$$

First, we calculate the eigenvalues of this matrix. The characteristic polynomial is

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) \tag{82}$$

Thus the eigenvalues are $\lambda_1 = -1, \lambda_2 = 2$. Since they are distinct, we can proceed with this method.

We know the solution for $x_1(t), x_2(t)$ is of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{-t} + c_1 e^{2t} \\ c_2 e^{-t} + c_3 e^{2t} \end{bmatrix} \tag{83}$$

At $t = 0$, we have $y(0) = 1$ and $\frac{dy}{dt}(0) = 1$. Using our differential equation (eq. (80)), we can get $\frac{d^2y}{dt^2}(0) = \frac{dy}{dt}(0) + 2y(0) = 3$. Plugging these in,

$$x_1(0) = y(0) = 1 = c_0 + c_1 \tag{84}$$

$$x_2(0) = \frac{dy}{dt}(0) = 1 = c_2 + c_3 \tag{85}$$

$$\frac{dx_1}{dt}(0) = \frac{dy}{dt}(0) = 1 = -c_0 + 2c_1 \tag{86}$$

$$\frac{dx_2}{dt}(0) = \frac{d^2y}{dt^2}(0) = 3 = -c_2 + 2c_3 \tag{87}$$

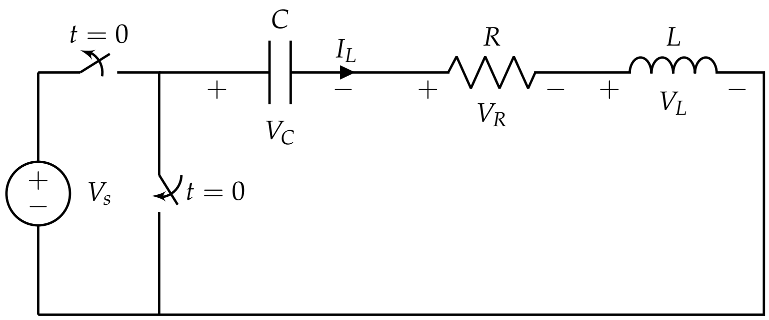
This gives $c_0 = \frac{1}{3}, c_1 = \frac{2}{3}, c_2 = -\frac{1}{3}, c_3 = \frac{4}{3}$. Alternatively, you could've seen that $c_2 = -c_0$ and $c_3 = 2c_1$ since $x_2(t)$ is the derivative of $x_1(t)$ which makes it solvable with just the first 2 equations. Thus we have

$$x_1(t) = y(t) = \frac{1}{3}e^{-t} + \frac{2}{3}e^{2t} \tag{88}$$

$$x_2(t) = \frac{dy(t)}{dt} = -\frac{1}{3}e^{-t} + \frac{4}{3}e^{2t} \tag{89}$$

5. RLC Responses

Consider the following circuit:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

The sequence of problems 2 - 6 combined will try to show you the various RLC system responses and how they relate to changing circuit properties.

(a) We first need to construct our state space system. Our state variables are the current through the inductor $x_1(t) = I_L(t)$ and the voltage across the capacitor $x_2(t) = V_C(t)$ since these are the quantities whose derivatives show up in the system of equations governing our circuit. Now, **show that the system of differential equations in terms of our state variables that describes this circuit for $t \geq 0$ is**

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) \quad (90)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \frac{1}{C}x_1(t). \quad (91)$$

Solution: For this part, we need to find two differential equations, each including a derivative of one of the state variables.

First, let's consider the capacitor equation $I_C(t) = C\frac{\mathrm{d}}{\mathrm{d}t}V_C(t)$. In this circuit, $I_C(t) = I_L(t)$, so we can write

$$I_C(t) = C\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = I_L(t) \quad (92)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = \frac{1}{C}I_L(t). \quad (93)$$

If we use the state variable names, we can write this as

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \frac{1}{C}x_1(t), \quad (94)$$

so now we have one differential equation.

For the other differential equation, consider the voltage drop across the capacitor, resistor and inductor. At $t \geq 0$, the voltage difference between the positive '+' terminal of C and the negative '-' terminal of L is given by

$$V_C(t) + V_R(t) + V_L(t) = 0. \quad (95)$$

Using Ohm's Law $V_R(t) = RI_L(t)$ and the inductor equation $V_L(t) = L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t)$, we can write this as

$$V_C(t) + RI_L(t) + L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = 0, \tag{96}$$

which we can rewrite as

$$\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = -\frac{R}{L}I_L(t) - \frac{1}{L}V_C(t). \tag{97}$$

If we use the state variable names, this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t), \tag{98}$$

and we have a second differential equation.

To summarize, the final system is

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) \tag{99}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \frac{1}{C}x_1(t). \tag{100}$$

(b) **Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix}$.** This should be in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

Solution: By inspection from the previous part, we have

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{101}$$

which is in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$, with

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}. \tag{102}$$

(c) **Show that, for the 2×2 matrix A, the two eigenvalues of A are**

$$\lambda_1 = -\frac{1}{2}\frac{R}{L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{103}$$

$$\lambda_2 = -\frac{1}{2}\frac{R}{L} - \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}. \tag{104}$$

(HINT: The quadratic formula will be involved.)

Solution: To find the eigenvalues, we'll solve $\det(A - \lambda I) = 0$. In other words, we want to find λ such that

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\frac{R}{L} - \lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{bmatrix}\right) \tag{105}$$

$$= -\lambda\left(-\frac{R}{L} - \lambda\right) + \frac{1}{LC} \tag{106}$$

$$= \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0. \tag{107}$$

The quadratic formula gives

$$\lambda = -\frac{1}{2}\frac{R}{L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{108}$$

as desired.

(d) **Under what condition on the circuit parameters R, L, C will A have two distinct real eigenvalues?**

Solution: For both eigenvalues to be real and distinct, we need the quantity inside the square root to be positive. In other words, we need

$$\frac{R^2}{L^2} - \frac{4}{LC} > 0, \tag{109}$$

or, equivalently,

$$R > 2\sqrt{\frac{L}{C}}. \tag{110}$$

(e) **Under what condition on the circuit parameters R, L, C will A have two imaginary eigenvalues? What will the eigenvalues be in this case?** **Solution:** The only way for both eigenvalues to be purely imaginary is to have $R = 0$. In this case, the eigenvalues would be

$$\lambda = \pm j\sqrt{\frac{1}{LC}}. \tag{111}$$

(f) Assuming that the circuit parameters are such that there are a pair of (potentially complex) eigenvalues λ_1, λ_2 so that $\lambda_1 \neq \lambda_2$, **show that the corresponding eigenvectors $\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}$ are**

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix} \quad \text{and} \quad \vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix}. \tag{112}$$

Solution: We use the definition of an eigenvector and eigenvalue. We want $A\vec{v}_{\lambda_i} = \lambda_i \vec{v}_{\lambda_i}$. Note that, for any y,

$$A\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{y}{L} \\ 0 \end{bmatrix} \tag{113}$$

is not a scalar multiple of $\begin{bmatrix} 0 \\ y \end{bmatrix}$, so no eigenvector is of the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$. Thus they must all be of the form $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ with $y_1 \neq 0$, and we can divide through by y_1 to show that every eigenvector is of the form $\begin{bmatrix} 1 \\ y \end{bmatrix}$ for some y.

Thus,

$$\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ y \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} \lambda_i \\ \lambda_i \cdot y \end{bmatrix} \tag{114}$$

We also know that:

$$\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} - \frac{y}{L} \\ \frac{1}{C} \end{bmatrix} \tag{115}$$

Equating the two equations from above gives:

$$\begin{bmatrix} \lambda_i \\ \lambda_i \cdot y \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} - \frac{y}{L} \\ \frac{1}{C} \end{bmatrix}. \tag{116}$$

From the second row we see that $y = \frac{1}{\lambda_i C}$. Now we find the eigenvectors as:

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix} \tag{117}$$

$$\vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix} \tag{118}$$

Alternatively, you can try to use the standard approach of finding the nullspace of $A - \lambda_i I$ to arrive at the same answer as above.

(g) Assuming circuit parameters such that the two eigenvalues of A are distinct, let $V = \begin{bmatrix} \vec{v}_{\lambda_1} & \vec{v}_{\lambda_2} \end{bmatrix}$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. **Show that the $\widetilde{A}$ so that $\frac{d}{dt}\vec{\widetilde{x}}(t) = \widetilde{A}\vec{\widetilde{x}}(t)$ is**

$$\widetilde{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{119}$$

(HINT: Write out the original differential equation $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$, then use the given change of coordinates to write everything in terms of $\vec{\widetilde{x}}(t)$.)

Solution: V is given by:

$$V = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix} \tag{120}$$

We know that V transforms from the $\widetilde{x}$ coordinate frame to the x coordinate frame, V^{-1} transforms back, and A takes gives the relationship from x to $\frac{d}{dt}x$.
Therefore to go from $\widetilde{x}$ to $\frac{d}{dt}\widetilde{x}$:

$$\widetilde{A} = V^{-1}AV \tag{121}$$

$$= \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix} \tag{122}$$

$$= \frac{\lambda_1 \lambda_2 C}{\lambda_1 - \lambda_2} \begin{bmatrix} \frac{1}{\lambda_2 C} & -1 \\ -\frac{1}{\lambda_1 C} & 1 \end{bmatrix} \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix} \tag{123}$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{124}$$

You didn't have to multiply things out explicitly. You could have just noticed that the eigenvector matrix will diagonalize the A matrix such that $AV = V\Lambda$ or $V^{-1}AV = \Lambda$, as per one of the problems on the last homework.

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