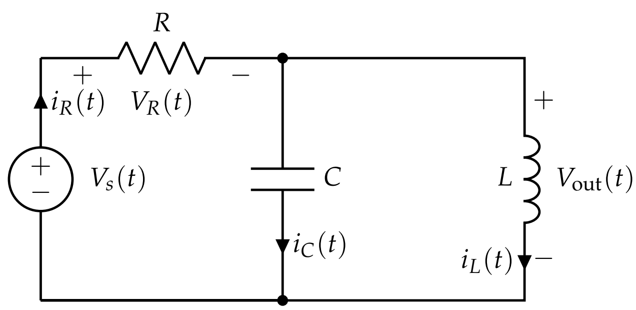
Homework 5

This homework is due on Friday, February 24, 2023, at 11:59PM. Self-grades and HW Resubmissions are due on the following Friday, March 3, 2023, at 11:59PM.

1. Phasor-Domain Circuit Analysis

The analysis techniques you learned previously in 16A for resistive circuits are equally applicable for analyzing circuits driven by sinusoidal inputs in the phasor domain. In this problem, we will walk you through the steps with a concrete example.

Consider the following circuit where the input voltage is sinusoidal. The end goal of our analysis is to find an equation for $V_{\text{out}}(t)$.



The components in this circuit are given by:

$$V_s(t) = 10\sqrt{2}\cos\left(100t - \frac{\pi}{4}\right) \tag{1}$$

$$R = 5\,\Omega \tag{2}$$

$$L = 50\,\text{mH} \tag{3}$$

$$C = 2\,\text{mF} \tag{4}$$

(a) **Give the amplitude V_0, input frequency ω, and phase ϕ of the input voltage V_s.**

Solution: A sinusoid takes the form $v(t) = V_0\cos(\omega t + \phi)$. Given $V_s(t)$, we find:

$$V_0 = 10\sqrt{2}\text{V} \tag{5}$$

$$\omega = 100\frac{\text{rad}}{\text{s}} \tag{6}$$

$$\phi = -\frac{\pi}{4}\text{rad} \tag{7}$$

(b) Transform the circuit into the phasor domain. **What are the impedances of the resistor, capacitor, and inductor? What is the phasor $\widetilde{V}_S$ of the input voltage $V_s(t)$?**

Solution:

$$Z_L = \mathrm{j}\omega L = \mathrm{j}5\Omega \tag{8}$$

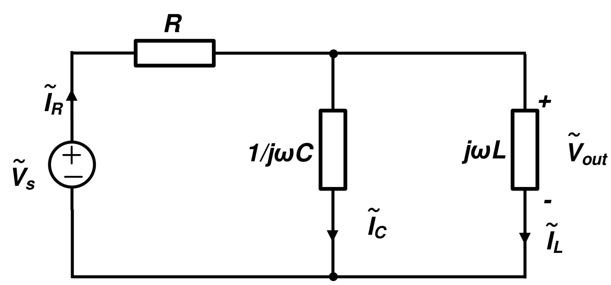
$$Z_C = \frac{1}{\mathrm{j}\omega C} = -\mathrm{j}5\Omega \tag{9}$$

$$Z_R = R = 5\Omega \tag{10}$$

$$\widetilde{V}_s = |V_s|e^{\mathrm{j}\angle V_s} = 10\sqrt{2}e^{-\mathrm{j}\frac{\pi}{4}} \tag{11}$$

(c) Use the circuit equations to **solve for** $\widetilde{V}_{\text{out}}$, the phasor representing the output voltage.

Solution: The phasor representation of the circuit is shown below:



Where

$$\widetilde{I}_R = \frac{\widetilde{V}_S - \widetilde{V}_{\text{out}}}{R} \tag{12}$$

$$\widetilde{I}_L = \frac{\widetilde{V}_{\text{out}}}{\mathrm{j}\omega L} \tag{13}$$

$$\widetilde{I}_C = \widetilde{V}_{\text{out}} \cdot \mathrm{j}\omega C \tag{14}$$

Rewriting the current relation in terms of voltage phasors gives:

$$\frac{\widetilde{V}_S - \widetilde{V}_{\text{out}}}{R} = \frac{\widetilde{V}_{\text{out}}}{\mathrm{j}\omega L} + \widetilde{V}_{\text{out}} \cdot \mathrm{j}\omega C \tag{15}$$

Substituting the component values in the above equation we get

$$\frac{\widetilde{V}_S - \widetilde{V}_{\text{out}}}{5} = \frac{\widetilde{V}_{\text{out}}}{5\mathrm{j}} + \frac{\widetilde{V}_{\text{out}} \cdot \mathrm{j}}{5} \tag{16}$$

$$= \frac{\widetilde{V}_{\text{out}}}{5\mathrm{j}} - \frac{\widetilde{V}_{\text{out}}}{5\mathrm{j}} \tag{17}$$

$$= 0 \tag{18}$$

Which gives:

$$\widetilde{V}_{\text{out}} = \widetilde{V}_S \tag{19}$$

We found that $\widetilde{V}_{\text{out}} = \widetilde{V}_S$ because this circuit is in resonance; i.e., the capacitor and inductor have the exact values that cause current and voltage to endlessly oscillate between them at this frequency. If we chose a different value for ω with these same component values, the circuit would not be in resonance and $\widetilde{V}_{\text{out}}$ and $\widetilde{V}_S$ would no longer be equal.

One may think that this answer seems weird. For $\widetilde{V}_{\text{out}}$ to equal $\widetilde{V}_S$ means that no current is flowing through the resistor. This means that somehow, the impedance of the parallel L and C combination would have to be infinity. Let's check what that is:

$$Z_L \parallel Z_C = \frac{(\text{j}5) \cdot (-\text{j}5)}{\text{j}5 + (-\text{j}5)} = +\infty \tag{20}$$

Wow! Indeed it is infinity. This shows something counterintuitive that can occur with phasors and impedances. For resistors, one may think that parallel connections always lower the resistance. However, since imaginary impedances can be positive imaginary and negative imaginary, a parallel connection can make the impedance bigger or smaller. The same kind of counterintuitive behavior is also possible for series combinations. Resistors in series always increase the resistance. But the same L and C in series can combine to have a zero impedance at the natural frequency.

If one wants to know why **something divided by 0** is ∞ in the complex plane, read this Wiki article: Riemann Sphere. This is another facet of complex analysis, and why engineers were drawn to it when modeling physical systems for design purposes.

2. Hambley P6.55

Consider the circuit shown in Figure 1. The input signal is given by

$$v_{\text{in}}(t) = 5 + 5\cos(2000\pi t) \tag{21}$$

Find an expression for the output $v_{\text{out}}(t)$ in steady-state conditions.

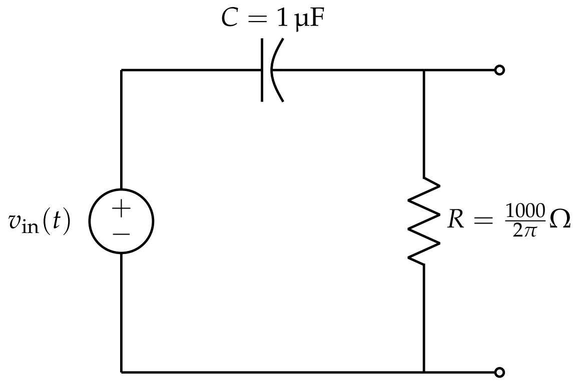


Figure 1: P6.55

(HINT: Use superposition. That is, find $v_{\text{out},1}(t)$ which is the output voltage if the input is $v_{\text{in},1}(t) = 5$, and then find $v_{\text{out},2}(t)$ which is the output voltage if the input is $v_{\text{in},2}(t) = 5\cos(2000\pi t)$. What is $v_{\text{out}}(t)$ in terms of $v_{\text{out},1}(t)$ and $v_{\text{out},2}(t)$?)

Solution: The input signal is given by

$$v_{\text{in}}(t) = 5 + 5\cos(2000\pi t) \tag{22}$$

We can find $\widetilde{V}_{\text{out}}$ as a function of ω and $\widetilde{V}_{\text{in}}$, then apply superposition. Applying the fact that $Z_C = \frac{1}{\mathrm{j}\omega C}$ and $Z_R = R$, we can use the voltage divider formula to write

$$\widetilde{V}_{\text{out}} = \frac{Z_R}{Z_R + Z_C}\widetilde{V}_{\text{in}} \tag{23}$$

$$= \frac{R}{R + \frac{1}{\mathrm{j}\omega C}}\widetilde{V}_{\text{in}} \tag{24}$$

$$= \frac{\mathrm{j}\omega RC}{\mathrm{j}\omega RC + 1}\widetilde{V}_{\text{in}} \tag{25}$$

Now, let's first consider $v_{\text{in},1}(t) = 5$ as per the hint. We have $\widetilde{V}_{\text{in},1} = 5e^{\mathrm{j}0}$, and the corresponding frequency is $\omega = 0$ (we can write $v_{\text{in},1}(t) = 5 = 5\cos(0 \cdot t + 0)$). Thus, plugging into eq. (25), we get $\widetilde{V}_{\text{out},1} = \frac{\mathrm{j}0}{\mathrm{j}0+1} = 0$. Hence, $v_{\text{out},1}(t) = 0$. Next, we consider $v_{\text{in},2}(t) = 5\cos(2000\pi t)$. We have $\widetilde{V}_{\text{in},2} = 5e^{\mathrm{j}0}$, and the corresponding frequency is $\omega = 2000\pi$. Thus, plugging into eq. (25), we get $\widetilde{V}_{\text{out},2} = \frac{\mathrm{j}(2000\pi)RC}{\mathrm{j}(2000\pi)RC+1} \cdot 5$. Plugging in for the provided values of R and C, we obtain $\widetilde{V}_{\text{out},2} = 5\frac{\mathrm{j}}{1+\mathrm{j}}$. To convert this to polar form, we compute the magnitude and phase as follows:

$$\left|\widetilde{V}_{\text{out},2}\right| = 5\left|\frac{\mathrm{j}}{1+\mathrm{j}}\right| = 5\frac{|\mathrm{j}|}{|1+\mathrm{j}|} = \frac{5}{\sqrt{2}} \tag{26}$$

$$\angle\widetilde{V}_{\text{out},2} = \angle\left(\frac{5\mathrm{j}}{1+\mathrm{j}}\right) = \angle(5\mathrm{j}) - \angle(1+\mathrm{j}) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \tag{27}$$

Thus, we can equivalently write $\widetilde{V}_{\text{out},2} = \frac{5}{\sqrt{2}} e^{j\frac{\pi}{4}}$. Converting back into time domain, we get $v_{\text{out},2}(t) = \frac{5}{\sqrt{2}} \cos\left(2000\pi t + \frac{\pi}{4}\right)$. Now, we can apply superposition to find $v_{\text{out}}(t)$. Since we decomposed the input voltage as a sum of its components, we have

$$v_{\text{out}}(t) = \underbrace{v_{\text{out},1}(t)}_{0} + \underbrace{v_{\text{out},2}(t)}_{\frac{5}{\sqrt{2}} \cos\left(2000\pi t + \frac{\pi}{4}\right)} = \frac{5}{\sqrt{2}} \cos\left(2000\pi t + \frac{\pi}{4}\right) \tag{28}$$

3. Hambley P6.74

Derive an expression for the resonant frequency of the circuit shown in Figure 2. We define the resonant frequency to be the frequency at which the impedance is purely real (no imaginary component).

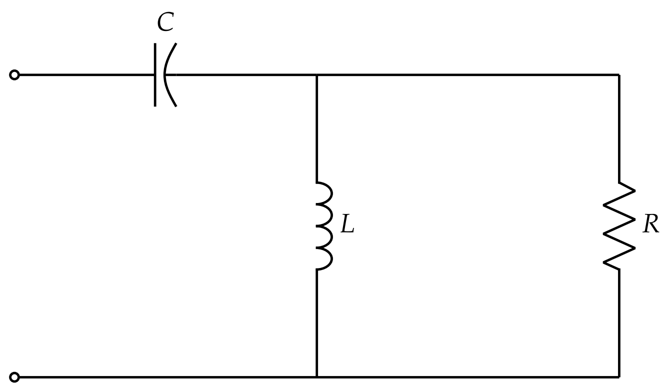


Figure 2: P6.74

Solution: The combined impedance is

$$Z = \frac{1}{j\omega C} + \frac{1}{\frac{1}{R} + \frac{1}{j\omega L}} \tag{29}$$

$$= -\frac{j}{\omega C} + \frac{\frac{1}{R} + j\frac{1}{\omega L}}{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L}\right)^2} \tag{30}$$

At the resonance frequency, the imaginary part must go to zero, so

$$\frac{\frac{1}{\omega L}}{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L}\right)^2} - \frac{1}{\omega C} = 0 \tag{31}$$

Solving for ω yields $\omega = \frac{1}{\sqrt{LC - \left(\frac{L}{R}\right)^2}}$.

4. Phasors and Eigenvalues

Suppose that we have the two-dimensional system of differential equations expressed in matrix/vector form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{32}$$

where for this problem, the matrix A and the vector $\vec{b}$ are both real.

(a) Give a necessary condition on the eigenvalues λ_k of A such that any impact of an initial condition will eventually completely die out. (i.e. the system will reach steady-state.) You don't have to prove this. *(HINT: Read Lecture 9 Slide 6.)*

Solution: The condition is that all eigenvalues must have real parts that are less than zero. In equations,

$$\forall k,\ \mathrm{Re}(\lambda_k) < 0 \tag{33}$$

This condition derives from the fact that the solutions to differential equations in the eigenspace contain terms that look like $e^{\lambda_k t}$. So, if all the eigenvalues are have strictly negative real parts, then all such exponential terms will die out.

If any of the eigenvalues have strictly positive real parts, then the exponential terms corresponding to them will blow up as growing exponentials.

The case of $\lambda = 0$ or having a zero real part in general (purely imaginary eigenvalues) is a little more ambiguous in feeling. This suggests that some constant offset (for the case of $\lambda = 0$) or some steady oscillation at a natural frequency of the system can persist throughout all time. But persisting isn't dying out and so we really want the eigenvalues to have strictly negative real parts for us to be able to ignore the initial conditions.

(b) Now assume that $u(t)$ has a phasor representation $\widetilde{U}$. In other words, $u(t) = \widetilde{U}e^{+j\omega t} + \overline{\widetilde{U}}e^{-j\omega t}$. Assume that the vector solution $\vec{x}(t)$ to the system of differential equations (32) can also be written in phasor form as

$$\vec{x}(t) = \vec{\widetilde{X}}e^{+j\omega t} + \overline{\vec{\widetilde{X}}}e^{-j\omega t}. \tag{34}$$

Derive an expression for $\vec{\widetilde{X}}$ involving $A, \vec{b}, j\omega, \widetilde{U}$, and the identity matrix I. (Here, we assume that $j\omega$, and $-j\omega$ are not eigenvalues of A, which indicates that $\det(j\omega I - A)$ and $\det(-j\omega I - A)$ are non-zero.)

(HINT: Plug eq. (34) into eq. (32) and simplify, using the rules of differentiation and grouping terms by which exponential $e^{\pm j\omega t}$ they multiply.

)

Solution: As the hint suggests, plugging back (34) into (32) we get the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\widetilde{X}}e^{j\omega t} + \overline{\vec{\widetilde{X}}}e^{-j\omega t}\right) = A(\vec{\widetilde{X}}e^{j\omega t} + \overline{\vec{\widetilde{X}}}e^{-j\omega t}) + \vec{b}(\widetilde{U}e^{+j\omega t} + \overline{\widetilde{U}}e^{-j\omega t}) \tag{35}$$

$$(j\omega\vec{\widetilde{X}}e^{j\omega t} - j\omega\overline{\vec{\widetilde{X}}}e^{-j\omega t}) = (A\vec{\widetilde{X}} + \vec{b}\widetilde{U})e^{j\omega t} + (A\overline{\vec{\widetilde{X}}} + \vec{b}\overline{\widetilde{U}})e^{-j\omega t} \tag{36}$$

$$\tag{37}$$

Note that $\tilde{\vec{X}}$ and $\tilde{U}$ do not depend on time since they are phasors. Next, we can group the coefficients with the same exponential terms,

$$\mathrm{j}\omega\tilde{\vec{X}} = A\tilde{\vec{X}} + \vec{b}\tilde{U} \tag{38}$$

$$-\mathrm{j}\omega\overline{\tilde{\vec{X}}} = A\overline{\tilde{\vec{X}}} + \vec{b}\overline{\tilde{U}} \tag{39}$$

$$\Rightarrow\overline{(\mathrm{j}\omega)}\overline{\tilde{\vec{X}}} = \overline{(A\tilde{\vec{X}} + \vec{b}\tilde{U})} \tag{40}$$

$$\Rightarrow(\mathrm{j}\omega)\tilde{\vec{X}} = (A\tilde{\vec{X}} + \vec{b}\tilde{U}) \tag{41}$$

We see that equations (38) and (41) match, which is good. Note that, here we are assuming A and $\vec{b}$ are real. Next, we can solve (38) to get $\tilde{\vec{X}}$:

$$\mathrm{j}\omega\tilde{\vec{X}} = A\tilde{\vec{X}} + \vec{b}\tilde{U} \tag{42}$$

$$\Rightarrow(\mathrm{j}\omega I - A)\tilde{\vec{X}} = \vec{b}\tilde{U} \tag{43}$$

$$\Rightarrow\tilde{\vec{X}} = (\mathrm{j}\omega I - A)^{-1}\vec{b}\tilde{U}. \tag{44}$$

Notice that we didn't need to explicitly deal with the conjugate terms. We know that their solution is just going to be the conjugate of what we computed here, because of the properties of complex arithmetic.

5. Circuit Design

In this problem, you will find a circuit where several components have been left *blank* for you to fill in.

Assume that the op-amp is *ideal*. A special note on op amps in frequency domain analysis: The op-amps you learned about in 16A can be used in exactly the same way for setting up differential equations and even Phasor analysis in 16B. Treat them as ideal op-amps and invoke the Golden Rules.

You have at your disposal *only one of each* of the following components (not including R_1):

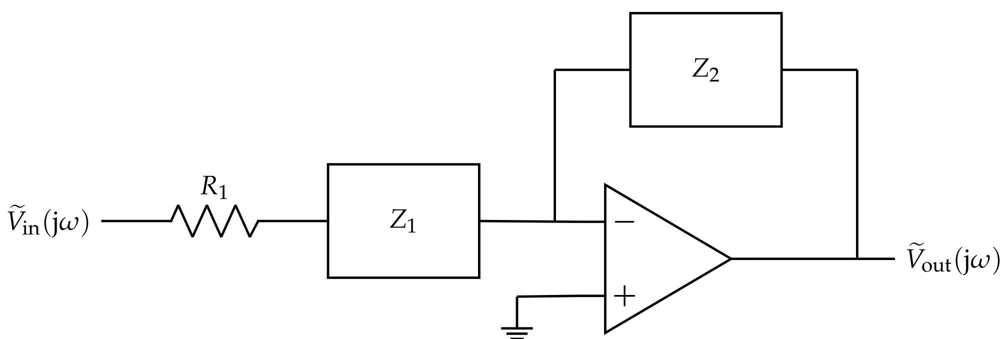
(a) an open circuit

(b) a short circuit

(c) a resistor (you choose from the values $R = 1\,\text{k}\Omega, 10\,\text{k}\Omega, 20\,\text{k}\Omega$)

(d) a capacitor (you choose from the values $C = 0.5\,\mu\text{F}, 1\,\mu\text{F}, 2\,\mu\text{F}$)

Consider the circuit below. The labeled voltages $\widetilde{V}_{\text{in}}(\text{j}\omega)$ and $\widetilde{V}_{\text{out}}(\text{j}\omega)$ are the phasor representations of $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$ respectively, where $v_{\text{in}}(t)$ has the form $v_{\text{in}}(t) = v_0 \cos(\omega t + \phi)$. The transfer function $H(\text{j}\omega)$ is defined as $H(\text{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\text{j}\omega)}{\widetilde{V}_{\text{in}}(\text{j}\omega)}$.



(a) Let $Z_1(\text{j}\omega)$ and $Z_2(\text{j}\omega)$ are the impedances of the boxes shown in the circuit diagram. **Write the expression of the transfer function $H(\text{j}\omega)$.**

Solution: The circuit is in the inverting amplifier configuration. Hence the transfer function is given by

$$H(\text{j}\omega) = -\frac{Z_2}{R_1 + Z_1} \tag{45}$$

(b) Let R_1 be $1\,\text{k}\Omega$. We have to find Z_1 and Z_2, such that the circuit's transfer function $H(\text{j}\omega)$ has the following properties:

- $|H(\text{j}0)| = 0$.
- $|H(\text{j}\infty)| = 10$.
- $|H(\text{j}10^3)| = \sqrt{50}$.

Using the fact that the circuit is a high pass filter, infer the components (we will find values later) of Z_1 and Z_2. Write the transfer function $H(\text{j}\omega)$ using these components.

(HINT: Try method of elimination: figure out what Z_2 cannot be. Once you find what Z_2 is, what does Z_1 have to be for the circuit to be a filter?)

Solution: The circuit should be a high-pass filter, so Z_2 cannot be a short circuit or a capacitor, otherwise $\widetilde{V}_{\text{out}}(\mathrm{j}\infty) = 0$. Also Z_2 cannot be open circuit as that will break the negative feedback. So Z_2 is a resistor, i.e. $Z_2 = R$.

Since Z_2 cannot be a capacitor, Z_1 must be a capacitor. Otherwise, we would not have any frequency dependent impedance in the circuit, which means it wouldn't be a filter. So $Z_1 = \frac{1}{\mathrm{j}\omega C}$. Hence the transfer function is

$$H(\mathrm{j}\omega) = -\frac{R}{R_1 + \frac{1}{\mathrm{j}\omega C}} \tag{46}$$

$$= -\frac{R}{R_1} \cdot \frac{1}{1 - \frac{\mathrm{j}}{\omega C R_1}} \tag{47}$$

Observe that $|H(\mathrm{j}0)| = 0$ and $|H(\mathrm{j}\infty)| = \frac{R}{R_1}$, so it is a high-pass filter.

(c) **Now use the facts that $|H(\mathrm{j}\infty)| = 10$ and $R_1 = 1\,\mathrm{k}\Omega$ to find the component value of Z_2.**

Solution: From $|H(\mathrm{j}\infty)| = 10 = \frac{R}{R_1}$, we know that $R = 10R_1 = 10\,\mathrm{k}\Omega$, which is one of the options for resistor values.

(d) **Finally use the fact that $|H(\mathrm{j}10^3)| = \sqrt{50}$ and the values of R_1 and Z_2 to find the component value of Z_1.**

Solution: We can now write the transfer function as

$$H(\mathrm{j}\omega) = -10\frac{1}{1 - \frac{\mathrm{j}}{1000\omega C}} \tag{48}$$

We get

$$|H(\mathrm{j}10^3)| = 10\frac{1}{\sqrt{1 + \frac{1}{(10^6 C)^2}}} = \sqrt{50} \tag{49}$$

$$\implies 100\frac{1}{1 + \frac{1}{(10^6 C)^2}} = 50 \tag{50}$$

$$\implies \frac{1}{1 + \frac{1}{(10^6 C)^2}} = \frac{1}{2} \tag{51}$$

$$\implies \frac{1}{(10^6 C)^2} = 1 \tag{52}$$

$$\implies C = 1\,\mu\mathrm{F} \tag{53}$$

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