
Homework 8

This homework is due on Friday, October 21, 2022, at 11:59PM. Self-grades and HW Resubmissions are due on Friday, October 28, 2022, at 11:59PM.

1. Motor Driver and System Identification

In the lab project, you will be designing SIXT33N, a mischievous little robot who *might* just do what you want — if you design it correctly. In phases 1 and 2, you will build the **legs** of SIXT33N: you will be designing SIXT33N's wheels and developing a linear model for the car system. The wheels will be driven by two 9-Volt DC motors whose driver control circuit is shown in Figure 1.

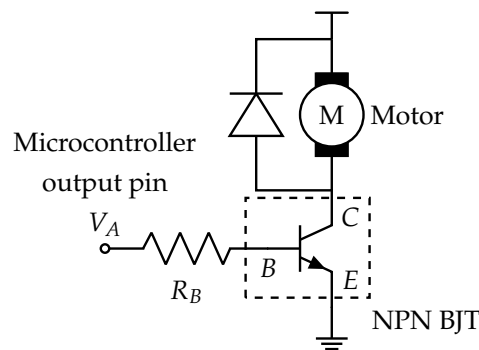


Figure 1: Motor Controller Circuit

There is some minimum voltage required to deliver enough power to the motors to overcome the static friction and start them, but after that point, we treat the motor speed as approximately **linear** with the applied voltage V_A (this will be the basis of the system model you will develop in this problem).

As it is difficult to use a microcontroller (MSP430 in hands-on lab; Arduino in lab sim) to generate a true adjustable DC signal, we will instead make use of its PWM function. A PWM, or pulse-width modulated, signal is a square wave with a variable duty cycle (the proportion of a cycle period for which the power source is turned on, or logic HIGH). PWM is used to digitally change the average voltage delivered to a load by varying the duty cycle. If the frequency is large enough, the on-off switching is imperceptible, but the average voltage delivered to the load changes proportionally with the duty cycle. Hence, changing the duty cycle corresponds to changing the DC voltage supplied to the motor. An example can be seen in figure 2.

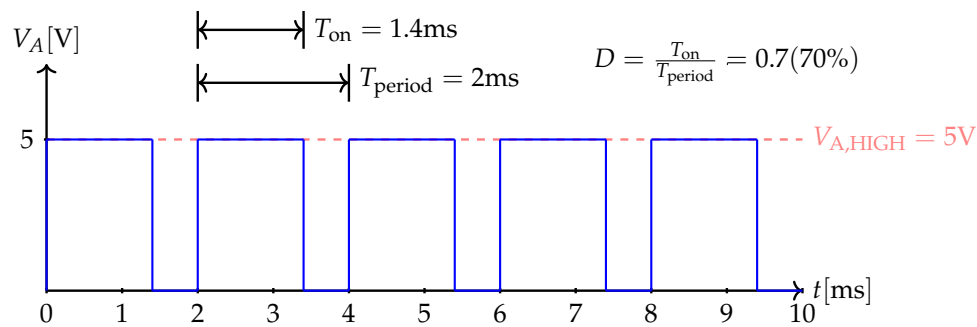
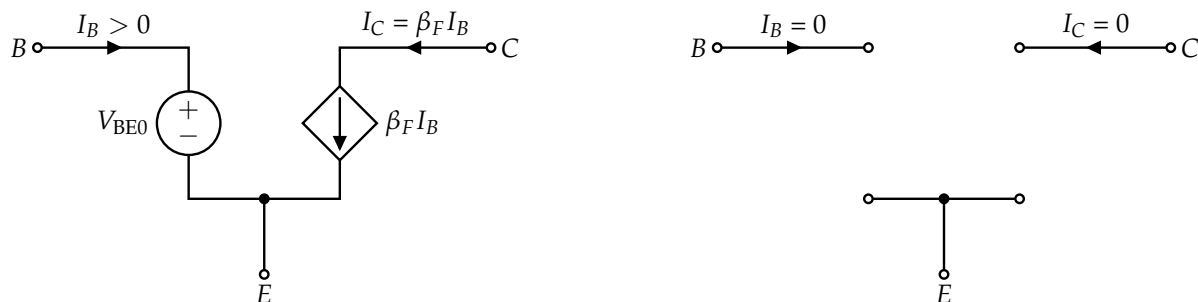


Figure 2: PWM Example with switching frequency 500Hz and 70% duty cycle

The PWM pin (V_A) is connected via a resistor (R_B) to the "Base (B)" of an NPN bipolar junction transistor (BJT). This transistor, in reality, behaves a bit differently from the NMOS with which you are familiar, but for this class, you may assume that it is functionally the same as an NMOS, behaving as a switch. On the BJT, the three terminals are analogous to those of an NMOS: the "Base (B)" is the gate, the "Collector (C)" is the drain, and the "Emitter (E)" is the source.

The BJT in Figure 1 is switching between **ON** and **OFF** modes when V_A is HIGH and LOW respectively. The model for both ON and OFF states are shown in Figure 3. When the BJT turns on, V_{BE} can be modeled as a fixed voltage source with voltage value V_{BE0} . In ON mode, there is a **Current Controlled Current Source** modeled between "Collector (C)" and "Emitter (E)", i.e., current at the "Collector (C)" is an amplified version of current at the "Base (B)" (notice that positive I_B has to flow into the "Base (B)" for the relation $I_C = \beta_F I_B$ to hold). β_F is called the **Common-Emitter Current Gain**.

The diode in parallel with the motor is needed because of the inductive characteristics of the motor. If the motor is on and V_A switches to LOW, the inductive behavior of the motor maintains the current and the diode provides the path to dissipate it as the BJT is turned off. When the BJT turns on, the diode is off so there is no current flow through the diode.



(a) Model of BJT in ON mode (when V_A is logic HIGH)

(b) Model of BJT in OFF mode (when V_A is logic LOW)

Figure 3: Model of NPN BJT in Different Modes

Please use $V_{BE0} = 0.8\text{ V}$, $\beta_F = 100$, $V_{A,HIGH} = 5\text{ V}$ and $V_{A,LOW} = 0\text{ V}$ for all following calculations.

Part 1: Circuit analysis to construct the system model

- (a) Draw the equivalent motor controller circuit when the BJT is ON by substituting in the BJT model from Fig. 3a into Fig. 1. Express I_B and I_C for $V_A = V_{A,HIGH}$ as a function of R_B .

Solution: First, we draw the equivalent circuit in figure 4 to analyze the quantities which will appear in it: I_B and I_C .

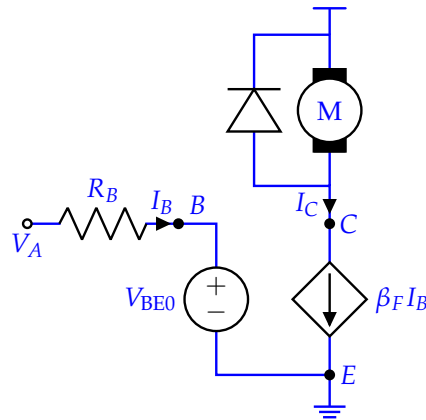


Figure 4: Equivalent circuit with model of BJT in ON state substituted in.

When V_A is HIGH, the voltage across the resistor in Figure 1 is $V_{A,HIGH} - V_{BE0}$. This voltage is what determines the the current flowing into the "Base(B)" (I_B) and current flowing into the "Collector (C)" (I_C).

$$I_B = \frac{V_{A,HIGH} - V_{BE0}}{R_B} = \frac{5\text{ V} - 0.8\text{ V}}{R_B} = \frac{4.2\text{ V}}{R_B} \quad (1)$$

$$I_C = \beta_F I_B = 100 \frac{4.2\text{ V}}{R_B} = \frac{420\text{ V}}{R_B} \quad (2)$$

Observe that downward directed current through the motor when V_A is HIGH is equal to the current I_C as the diode will not allow current in the direction from the motor voltage source to the collector (C).

- (b) Draw the equivalent motor controller circuit when the BJT is OFF by substituting in the BJT model from Fig. 3b into Fig. 1. Express I_B and I_C for $V_A = V_{A,LOW}$ as a function of R_B .

Solution: We draw the equivalent circuit as before.

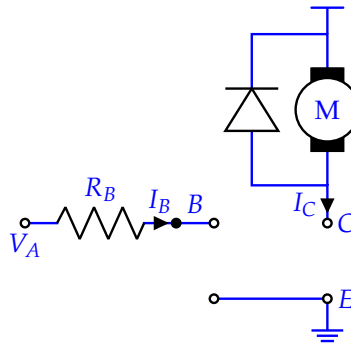


Figure 5: Equivalent circuit with model of BJT in OFF state substituted in.

When V_A is LOW, the BJT is OFF. As shown in Figure 5, the BJT is open from B to E , and from C to E . Thus $I_B = 0$ and $I_C = 0$.

Consider that while I_C can be zero, the current through the motor may not be as the diode conductive path allows for the inductor current of the motor to keep flowing. If we model the motor and diode as a circuit with a small resistor, R , and inductor, L , in series, we see that for a small R , we will have a large time constant $\tau = \frac{L}{R}$, which means that the current does not decay very much through the motor during the time the BJT is in the OFF state.

- (c) **Derive the average collector current, I_{AVG} , over one period, T_{period} , of the PWM signal, V_A , as a function of R_B and the duty cycle, D , of the PWM signal.** Hint: The time average of some signal $f(t)$ from time t_0 to t_1 is given as $f_{AVG} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} f(\tau) d\tau$. Figure 2 may be useful.

Solution: We compute the time average of I_C from $t = 0$ to $t = T_{\text{period}}$, where T_{period} is the period of the PWM voltage.

$$I_{AVG} = \frac{1}{T_{\text{period}} - 0} \int_0^{T_{\text{period}}} I_C(\tau) d\tau \quad (3)$$

Since I_C depends on V_A , and V_A switches between two values $V_{A,HIGH}$ and $V_{A,LOW}$, we will have corresponding values of current $I_{C,ON}$ and $I_{C,OFF}$. If we start at time 0, we spend times $0 \leq t \leq T_{on}$ with constant current $I_{C,ON}$ due to constant voltage $V_A = V_{A,HIGH}$. This first current $I_{C,ON}$ is the current we calculated in part (a). At times $T_{on} \leq t \leq T_{\text{period}}$, we turn our voltage V_A to its low value, with corresponding constant current $I_{C,OFF}$. This is the current we calculated in part (b). Our integral becomes the following:

$$I_{AVG} = \frac{1}{T_{\text{period}}} \int_0^{T_{\text{period}}} I_C(\tau) d\tau \quad (4)$$

$$= \frac{1}{T_{\text{period}}} \left(\int_0^{T_{on}} I_{C,ON} d\tau + \int_{T_{on}}^{T_{\text{period}}} I_{C,OFF} d\tau \right) \quad (5)$$

$$= \frac{1}{T_{\text{period}}} \left(I_{C,ON}(T_{on} - 0) + I_{C,OFF}(T_{\text{period}} - T_{on}) \right) \quad (6)$$

$$= I_{C,ON} \frac{T_{on}}{T_{\text{period}}} + I_{C,OFF} \frac{T_{\text{period}} - T_{on}}{T_{\text{period}}} \quad (7)$$

$$= I_{C,ON} D + 0 A(1 - D) \quad (8)$$

$$= I_{C,ON} D \quad (9)$$

$$= \frac{420 \text{ V}}{R_B} D \quad (10)$$

- (d) **In the previous part, explain briefly why is it sufficient to take the average over only one period if we are actually interested in the average collector current over multiple periods?**

Solution: It is enough to take the average over one period as this would be the same as the average that would occur over an large integer number of periods and close enough to the average for a large non-integer number of periods. Reasoning of this form is enough.

What follows is a mathematical statement of the intuition above for why we can look at only one period. It is not required but is here for your understanding. The assumption of having a high

PWM frequency is relevant here, as a high frequency implies a small period. We should have our period T be small relative to the time we set a certain duty cycle value, so that the time spent in a fraction of a cycle at the very end of our string of full periods does not contribute much to the average and will be negligible. In notation, let the time we average over be from $t = 0$ to $t = T = NT_{\text{period}} + \delta$, where N is an integer and δ is a time interval that is mid-cycle and not through the full period, i.e. $0 \leq \delta \leq T_{\text{period}}$.

$$I_{\text{AVG}} = \frac{1}{T} \int_0^T I_C(t) d\tau \quad (11)$$

$$= \frac{1}{T} \left(\sum_{i=0}^{N-1} \left(\int_{iT_{\text{period}}}^{(i+1)T_{\text{period}}} I_C(t) d\tau \right) + \int_{NT_{\text{period}}}^{NT_{\text{period}}+\delta} I_C(t) d\tau \right) \quad (12)$$

$$= \frac{1}{T} \left(\sum_{i=0}^{N-1} I_{C,\text{ON}} D + I_{C,\text{ON}} \min(\delta, T_{\text{on}}) \right) \quad (13)$$

$$= \frac{1}{T} (NI_{C,\text{ON}}T_{\text{on}} + I_{C,\text{ON}} \min(\delta, T_{\text{on}})) \quad (14)$$

$$= \frac{NT_{\text{on}}}{NT_{\text{period}} + \delta} I_{C,\text{ON}} + \frac{\min(\delta, T_{\text{on}})}{NT_{\text{period}} + \delta} I_{C,\text{ON}} \quad (15)$$

The minimum term comes from that the time mid-cycle δ may be less than or greater than T_{on} , so we may have current on for a shorter interval or have an interval in which we have turned our current off after spending the full time T_{on} with the current on. However, notice that as we make N large, which is to say that our time T we are averaging over is large relative to our period T_{period} , our average value approaches $\frac{T_{\text{on}}}{T_{\text{period}}} I_{C,\text{ON}} = DI_{C,\text{ON}}$, where we have our duty cycle. This is what allows our duty cycle to determine our average current and voltage value, and why we should have a high enough PWM frequency for the system in consideration. If we were to only hold our duty cycle for a short period of time, on par with the period of the PWM signal, we lose this dependence of the average current on duty cycle and would have behavior that is a function of the time we hold our duty cycle for. This is an example of the important idea in control and computing that the parameters of a system (e.g. R , C in transistors, PWM frequency here) are chosen in such a way that trades off performance (usually speed) and reliability.

- (e) If $R_B = 2 \text{ k}\Omega$, what is the average collector current, I_{AVG} , that drives the motor when the duty cycle of the PWM signal is equal to 25%?

Solution: From part (c), we have that $I_{\text{AVG}} = \frac{420\text{V}}{R_B} D$. For $R_B = 2 \text{ k}\Omega$ and $D = 0.25$ (25%), $I_{\text{AVG}} = \frac{420\text{V}}{2\text{k}\Omega} 0.25 = 52.5 \text{ mA}$.

Part 2: Learning a Car Model from data

To control the car, we need to build a model of the car first. Instead of designing a complex nonlinear model, we will approximate the system with a linear model to work for small perturbations around an equilibrium point. The following model applies separately to each wheel (and associated motor) of the car:

$$v_L[i] = \theta_L u_L[i] - \beta_L \quad (16)$$

$$v_R[i] = \theta_R u_R[i] - \beta_R \quad (17)$$

Notice that this particular model has no state variables since we are measuring velocity directly here. To do system ID, we decide to use the exact same input $u_L[i] = u_R[i] = u[i]$ for both motors. We measure both velocities however.

Meet the variables at play in this model: (**Note: the β here have nothing to do with the previous part.**)

- i - The current timestep of the model. Since we model the car as a discrete-time system, n will advance by 1 on every new sample in the system.
- $v_L[i]$ - The discrete-time velocity (in units of ticks/timestep) of the left wheel, reading from the motor.
- $v_R[i]$ - The discrete-time velocity (in units of ticks/timestep) of the right wheel, reading from the motor.
- $u[i]$ - The input to each wheel. The duty cycle of the PWM signal (V_A), which is the percentage of the square wave's period for which the square wave is HIGH, is mapped to the range $[0, 255]$. Thus, $u[i]$ takes a value in $[0, 255]$ representing the duty cycle. For example, when $u[i] = 255$, the duty cycle is 100 %, and the motor controller just delivers a constant signal at the system's HIGH voltage, delivering the maximum possible power to the motor. When $u[i] = 0$, the duty cycle is 0 %, and the motor controller delivers 0 V. The duty cycle (D) can be written as

$$\text{duty cycle (D)} = \frac{u[i]}{255} \quad (18)$$

- $\theta(\theta_L, \theta_R)$ - Relates change in input to change in velocity. **Its units are ticks/(timestep · duty cycle)**. Since our model is linear, we assume that θ is the same for every unit increase in $u[i]$. This is empirically measured using the car. **You will have a separate θ for your left and your right wheel**(θ_L, θ_R).
 - $\beta(\beta_L, \beta_R)$ - Similarly to θ , β is dependent upon many physical phenomena, so we will empirically determine it using the car. β represents a constant offset in the velocity of the wheel, and hence **its units are ticks/timestep**. Note that you will also typically have a different β for your left and your right wheel (i.e. $\beta_L \neq \beta_R$). **These β_L and β_R are different from the β_F of the transistor.**
- (f) By measuring the car with a PWM signal at different duty cycles, we can collect the velocity data of the left and right wheel, as shown in the following table:

Table 1: The velocity of the left and the right wheel at different duty cycles of PWM signal

Duty Cycle $\times 255$ ($u[i]$)	Velocity of the left wheel ($v_L[i]$)	Velocity of the right wheel ($v_R[i]$)
80	147	127
120	218	187
160	294	253
200	370	317

Since the same input is applied to both the wheels, we can take advantage of the same “horizontal stacking” trick you’ve seen before to be able to reuse computation. To identify the system we need to setup matrix equations of left and right wheel in the form of:

$$D_{\text{data}}P \approx S \quad (19)$$

where $P = \begin{bmatrix} \theta_L & \theta_R \\ \beta_L & \beta_R \end{bmatrix}$. **Find the matrix D_{data} and matrix S needed to perform system identification to get the matrix of parameters of the left and right wheel, P .**

Solution: We can relate our data points in our table using the model equations eqs. (16) and (17). While we do not know the time indices, we can call them i_1 to i_4 . We start with the left wheel speed data. Approximate equality is used due to the possible presence of noise or unmodelled behavior.

$$v_L[i_1] \approx \theta_L u[i_1] - \beta_L \quad (20)$$

$$v_L[i_2] \approx \theta_L u[i_2] - \beta_L \quad (21)$$

$$v_L[i_3] \approx \theta_L u[i_3] - \beta_L \quad (22)$$

$$v_L[i_4] \approx \theta_L u[i_4] - \beta_L \quad (23)$$

Putting the left and right hand sides into vectors, we get the following equation.

$$\underbrace{\begin{bmatrix} v_L[i_1] \\ v_L[i_2] \\ v_L[i_3] \\ v_L[i_4] \end{bmatrix}}_{\vec{s}_L} \approx \begin{bmatrix} \theta_L u[i_1] - \beta_L \\ \theta_L u[i_2] - \beta_L \\ \theta_L u[i_3] - \beta_L \\ \theta_L u[i_4] - \beta_L \end{bmatrix} = \begin{bmatrix} u[i_1] & -1 \\ u[i_2] & -1 \\ u[i_3] & -1 \\ u[i_4] & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \theta_L \\ \beta_L \end{bmatrix}}_{\vec{p}_L} \quad (24)$$

Here, \vec{s}_L is defined as the vector of measured wheel velocities, and \vec{p}_L is the vector of parameters for the left wheel. We will have corresponding vectors \vec{s}_R and \vec{p}_R for the right wheel when writing a similar set of equations with $v_R[i]$, θ_R , and β_R . These equations utilize the same values of $u[i]$.

$$\underbrace{\begin{bmatrix} v_R[i_1] \\ v_R[i_2] \\ v_R[i_3] \\ v_R[i_4] \end{bmatrix}}_{\vec{s}_R} \approx \begin{bmatrix} u[i_1] & -1 \\ u[i_2] & -1 \\ u[i_3] & -1 \\ u[i_4] & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \theta_R \\ \beta_R \end{bmatrix}}_{\vec{p}_R} \quad (25)$$

Since the matrix multiplying the parameter vectors are the same, we can concatenate \vec{s}_L, \vec{s}_R and \vec{p}_L, \vec{p}_R horizontally in the following way.

$$\underbrace{\begin{bmatrix} | & | \\ \vec{s}_L & \vec{s}_R \\ | & | \end{bmatrix}}_S \approx \underbrace{\begin{bmatrix} u[i_1] & -1 \\ u[i_2] & -1 \\ u[i_3] & -1 \\ u[i_4] & -1 \end{bmatrix}}_{D_{\text{data}}} \underbrace{\begin{bmatrix} | & | \\ \vec{p}_L & \vec{p}_R \\ | & | \end{bmatrix}}_P \quad (26)$$

Substituting in the numerical values we have our D_{data} and S matrices.

$$D_{\text{data}} = \begin{bmatrix} 80 & -1 \\ 120 & -1 \\ 160 & -1 \\ 200 & -1 \end{bmatrix} \quad S = \begin{bmatrix} 147 & 127 \\ 218 & 187 \\ 294 & 253 \\ 370 & 317 \end{bmatrix} \quad (27)$$

- (g) **Solve the matrix equation $D_{\text{data}}P \approx S$ with least squares to find $\theta_L, \theta_R, \beta_L,$ and β_R .** You may use a jupyter notebook for computation.

Solution: Our estimate of our parameter matrix is given by $\hat{P} = (D_{\text{data}}^\top D_{\text{data}})^{-1} D_{\text{data}}^\top S$. Using a computing tool, we can compute the matrix product or use a built-in least squares function (e.g. NumPy's `np.linalg.lstsq`). We get the following parameter value estimates: $\hat{\theta}_L = 1.862$, $\hat{\theta}_R = 1.59$, $\hat{\beta}_L = 3.5$, and $\hat{\beta}_R = 1.6$.

- (h) In most advanced systems, we usually use a combination of a physics-based equation and a data-centric approach to build the model. In our case, the velocity of the motor can be written as

$$v[i] = kI_{\text{AVG}}(u[i]) - \beta \quad (28)$$

where $I_{\text{AVG}}(u[i])$ is the average collector current which is the function of the duty cycle that you have already derived in Part 1. k represents the response of your motor speed to the average current. In our simplified motor driver model in Part 1, you have already derived the expression for the I_{AVG} of the motor as a function of the circuit parameters and the duty cycle D . **If we assume that the model from Part 1 holds, determine the resistance ratio $\left(\frac{R_{B,\text{left}}}{R_{B,\text{right}}}\right)$ from the model parameters you identified in Part 2 item (f).** Assume that the left motor and the right motor respond the same, that is, $k_L = k_R$. The only difference is presumed to come from the resistors used.

Solution: First, we investigate how the current affects our model by starting with the velocity equation involving I_{AVG} for just one wheel.

$$v[i] = kI_{\text{AVG}}(u[i]) - \beta \quad (29)$$

$$= k \frac{420 \text{ V}}{R_B} D(u[i]) - \beta \quad (30)$$

After substituting in our average current expression from part (c), the only quantity that can be a function of $u[i]$ is our duty cycle, D . We recall from the introduction to part 2 that $D = \frac{u[i]}{255}$.

$$v[i] = k \frac{420 \text{ V}}{R_B} \left(\frac{u[i]}{255} \right) - \beta \quad (31)$$

$$= \left(k \frac{420 \text{ V}}{255 R_B} \right) u[i] - \beta \quad (32)$$

From our earlier learned model equations in eqs. (16) and (17), we can match coefficients and say that $\theta = k \frac{420 \text{ V}}{255 R_B}$. Since we identified $\hat{\theta}_L = 1.862$ and $\hat{\theta}_R = 1.59$ in part (f), we can take their ratio to see common terms cancel out.

$$\frac{\hat{\theta}_R}{\hat{\theta}_L} \approx \frac{k_R \frac{420 \text{ V}}{255 R_{B,\text{right}}}}{k_L \frac{420 \text{ V}}{255 R_{B,\text{left}}}} = \frac{R_{B,\text{left}}}{R_{B,\text{right}}} \quad (33)$$

$$\frac{1.59}{1.862} \approx \frac{R_{B,\text{left}}}{R_{B,\text{right}}} \quad (34)$$

$$0.85 \approx \frac{R_{B,\text{left}}}{R_{B,\text{right}}} \quad (35)$$

- (i) In order for the car to drive straight, the wheels must be moving at the same velocity. However, the data from Table 1 tell us that two motors cannot run at the same velocity if the duty cycles of driving PWM signals are the same. **Based on the model you extracted in Part 2 item (f), if we want the car to drive straight and $u_L = 100$, what should u_R be?**

Solution: To have the car drive straight, we want to find u_R satisfying $v_L = v_R$ for $u_L = 100$. From our identified constants, we can write the following equations.

$$v_L = 1.862u_L - 3.5 \quad (36)$$

$$v_R = 1.59u_R - 1.6 \quad (37)$$

Setting v_L and v_R equal to each other with $u_L = 100$, we get the following value of u_R .

$$1.862(100) - 3.5 = 1.59u_R - 1.6 \implies u_R \approx 115.91 \approx 116 \quad (38)$$

This observation was not required, but let us interpret the results of (f) and this part, (h). The significance of this calculation based on our learned parameters is that this difference in the required u_L and u_R required to have the same wheel velocities is a reflection of the underlying physics of our system. Taking the ratio $\frac{u_L}{u_R} = \frac{100}{116} \approx 0.86$, we find that we have nearly the same ratio as in (f). This is not a coincidence. u_L and u_R are proportional to the average voltages supplied, $V_{A,L,AVG}$ (left average voltage) and $V_{A,R,AVG}$ (right average voltage). If the resistance $R_{B,\text{right}}$ is larger than $R_{B,\text{left}}$ per part (f), this implies for the same average voltage applied to both motors, we will see a lower average current driving the right motor, thereby making its speed lower, assuming $k_L = k_R$ in the physical model equation. Note however, the ratios are not exactly equal for the reason of the β_L and β_R . However, because they are small relative to the larger values of $\theta_L u_L$ and $\theta_R u_R$ we see that we will have approximate equality, $\frac{u_L}{u_R} \approx \frac{R_{B,\text{left}}}{R_{B,\text{right}}}$, to move straight with $u_L = 100$.

This is to say that in our systems we will always have to account for the variation in the subparts of our system that are not necessarily identical, matched, or ideal. We made an assumption in part (f) that $k_L = k_R$, and this could introduce another adjustment required.

2. Stability Criterion

Consider the complex plane below, which is broken into non-overlapping regions A through H. The circle drawn on the figure is the unit circle $|\lambda| = 1$.

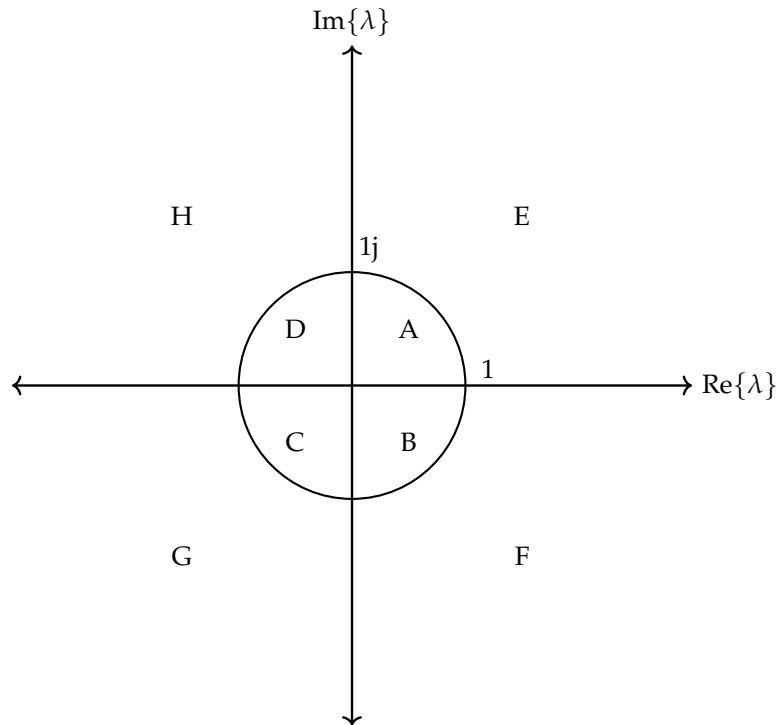


Figure 6: Complex plane divided into regions.

Consider the continuous-time system $\frac{d}{dt}x(t) = \lambda x(t) + v(t)$ and the discrete-time system $y[i+1] = \lambda y[i] + w[i]$. Here $v(t)$ and $w[i]$ are both disturbances to their respective systems.

In which regions can the eigenvalue λ be for the system to be *stable*? Fill out the table below to indicate *stable* regions. Assume that the eigenvalue λ does not fall directly on the boundary between two regions.

	A	B	C	D	E	F	G	H
Continuous Time System $x(t)$	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
Discrete Time System $y[i]$	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>

Solution: For the continuous time system to be stable, we need the real part of λ to be less than zero. Hence, C, D, G, H satisfy this condition.

On the other hand, for the discrete time system to be stable, we need the norm of λ to be less than one. Hence, A, B, C, D satisfy this condition.

3. Bounded-Input Bounded-Output (BIBO) Stability

BIBO stability is a system property where bounded inputs lead to bounded outputs. It's important because we want to certify that, provided our system inputs are bounded, the outputs will not "blow up". In this problem, we gain a better understanding of BIBO stability by considering some simple continuous and discrete systems, and showing whether they are BIBO stable or not.

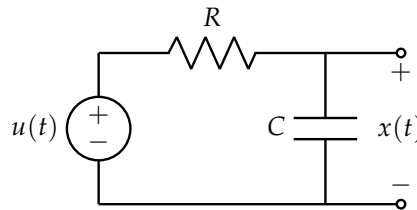
Recall that for the following simple scalar differential equation, we have the corresponding solution:

$$\frac{d}{dt}x(t) = ax(t) + bu(t) \quad x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau) d\tau. \quad (39)$$

And for the following discrete system, we have the corresponding solution:

$$x[i+1] = ax[i] + bu[i] \quad x[i] = a^i x[0] + \sum_{k=0}^{i-1} a^k bu[i-1-k] \quad (40)$$

(a) Consider the circuit below with $R = 1\Omega$, $C = 0.5F$. Let $x(t)$ be the voltage over the capacitor.



This circuit can be modeled by the differential equation

$$\frac{d}{dt}x(t) = -2x(t) + 2u(t) \quad (41)$$

Intuitively, we know that the voltage on the capacitor can never exceed the (bounded) voltage from the voltage source, so this system is BIBO stable. **Show that this system is BIBO stable, meaning that $x(t)$ remains bounded for all time if the input $u(t)$ is bounded. Equivalently, show that if we assume $|u(t)| < \epsilon$, $\forall t \geq 0$ and $|x(0)| < \epsilon$, then $|x(t)| < M$, $\forall t \geq 0$ for some positive constant M .** Thinking about this helps you understand what bounded-input-bounded-output stability means in a physical circuit.

(HINT: eq. (39) may be useful. You may want to write the expression for $x(t)$ in terms of $u(t)$ and $x(0)$ and then take the norms of both sides to show a bound on $|x(t)|$. Remember that norm in 1D is absolute value. Some helpful formulas are $|ab| = |a||b|$, the triangle inequality $|a + b| \leq |a| + |b|$, and the integral version of the triangle inequality $\left| \int_a^b f(\tau) d\tau \right| \leq \int_a^b |f(\tau)| d\tau$, which just extends the standard triangle inequality to an infinite sum of terms.)

Solution:

Using eq. (39), we get the solution to the scalar differential equation as

$$x(t) = e^{-2t}x(0) + \int_0^t e^{-2(t-\tau)}2u(\tau) d\tau. \quad (42)$$

Then we can try to bound $x(t)$ for $t \geq 0$. We first use the triangle inequality ($|a + b| \leq |a| + |b|$) to get

$$|x(t)| = \left| e^{-2t}x(0) + \int_0^t e^{-2(t-\tau)}2u(\tau) d\tau \right| \quad (43)$$

$$|x(t)| \leq \left| e^{-2t}x(0) \right| + \left| \int_0^t e^{-2(t-\tau)} 2u(\tau) d\tau \right| \quad (44)$$

We then use the property that the integral of absolute value will always be greater than the absolute value of the integral (equation (44) to (45)), and that an exponential is always positive (equation (45) to (46)):

$$|x(t)| \leq \left| e^{-2t}x(0) \right| + \int_0^t \left| e^{-2(t-\tau)} 2u(\tau) \right| d\tau \quad (45)$$

$$= e^{-2t}|x(0)| + \int_0^t e^{-2(t-\tau)} 2|u(\tau)| d\tau \quad (46)$$

Finally, plugging in our bounds for $|u(\tau)|$ and $|x(0)|$ and doing the integral:

$$|x(t)| \leq e^{-2t}\epsilon + \int_0^t e^{-2(t-\tau)} 2\epsilon d\tau \quad (47)$$

$$= e^{-2t}\epsilon + 2\epsilon e^{-2t} \int_0^t e^{2\tau} d\tau \quad (48)$$

$$= e^{-2t}\epsilon + 2\epsilon e^{-2t} \frac{1}{2} (e^{2t} - 1) \quad (49)$$

$$= e^{-2t}\epsilon + \epsilon (1 - e^{-2t}) \quad (50)$$

$$= \epsilon, \forall t \geq 0 \quad (51)$$

So we see that our state's magnitude is bounded for all time. Note that the negative exponent of the exponential is what makes this system stay bounded.

- (b) Assume $x(0) = 0$. **Show that the system eq. (39) is BIBO unstable when $a = j2\pi$ by constructing a bounded input that leads to an unbounded $x(t)$.**

It can be shown that the system eq. (39) is unstable for any purely imaginary a by a similar construction of a bounded input.

Solution: Recall the solution of $x(t)$ with the initial condition at zero

$$x(t) = \int_0^t e^{a(t-\tau)} b u(\tau) d\tau. \quad (52)$$

Remember, the style of argumentation here is the "counterexample" style. The question asks you to show that *some* bounded input exists that will make the state grow without bound.

Because we know we can get an integral to diverge if we are just integrating a nonzero constant, we decide to try the bounded input $u(t) = \epsilon e^{j2\pi t}$, whose magnitude is equal to ϵ for all t .

Plugging this input and a value in, we see

$$x(t) = \int_0^t e^{j2\pi(t-\tau)} b \epsilon e^{j2\pi\tau} d\tau = \int_0^t e^{j2\pi t} b \epsilon d\tau. \quad (53)$$

Factoring out the terms that do not depend on τ , we are left with

$$x(t) = b \epsilon e^{j2\pi t} \int_0^t d\tau. \quad (54)$$

Solving this integral, we get

$$x(t) = b \epsilon t e^{j2\pi t}. \quad (55)$$

Now taking the magnitude of $x(t)$ using the fact that $|e^{j\omega t}| = 1$ for all ω , we get $|x(t)| = \epsilon |b| t$ which clearly diverges as $t \rightarrow \infty$.

- (c) Consider the discrete-time system and its solution in eq. (40). **Show that if $|a| > 1$, then even if $x[0] = 0$, a bounded input can result in an unbounded output, i.e. the system is BIBO unstable.** (HINT: The formula for the sum of a geometric sequence may be helpful.)

Solution: Consider when $x[0] = 0$ and $u[i] = 1 \forall i$. This gives

$$x[i] = a^i x[0] + \sum_{k=0}^{i-1} a^k b u[i-1-k] \quad (56)$$

$$= \sum_{k=0}^{i-1} a^k b \quad (57)$$

$$= b \frac{a^i - 1}{a - 1} \quad (\text{as this is the sum of a geometric series}) \quad (58)$$

When $|a| > 1$, then a^i has magnitude that grows without bound, and thus $|x[i]|$ does as well. We also know this from the convergence criteria for geometric series; when the common ratio $a > 1$, the series does not converge to a finite number as $i \rightarrow \infty$.

- (d) Consider the discrete-time system

$$x[i+1] = -3x[i] + u[i]. \quad (59)$$

Is this system stable or unstable? Give an initial condition $x(0)$ and a sequence of non-zero inputs for which the state $x[i]$ will always stay bounded. (HINT: See if you can find any input pattern that results in an oscillatory behavior.)

Solution:

The system is unstable since the eigenvalue -3 has magnitude ≥ 1 . To see this more explicitly, any non-zero $x[0]$ and (bounded) $u[i] = 0 \forall i \in \mathbb{N}$ will lead to unbounded x .

Consider $x[0] = 0$ and the input $u[i] = 1, 3, 1, 3, 1, 3, \dots$

t	0	1	2	3	\dots
$x[i]$	0	1	0	1	\dots
$u[i]$	1	3	1	3	\dots
$-3x[i] + u[i]$	1	0	1	0	\dots

In this case, we get $x[i] = 0$ when t is even, and $x[i] = 1$ when i is odd. In fact, there are an infinite number of input sequences that would result in bounded outputs.

4. Eigenvalue Placement through State Feedback

Consider the following discrete-time linear system:

$$\vec{x}[i+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]. \quad (60)$$

In standard language, we have $A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the form: $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$.

(a) **Is this discrete-time linear system stable in open loop (without feedback control)?**

Solution: We have to calculate the eigenvalues of matrix A . Thus,

$$0 = \det(\lambda I - A) \quad (61)$$

$$= \det \begin{bmatrix} \lambda + 2 & -2 \\ 2 & \lambda - 3 \end{bmatrix} \quad (62)$$

$$= \lambda^2 - \lambda - 2 \quad (63)$$

$$\implies \lambda_1 = 2, \quad \lambda_2 = -1 \quad (64)$$

Since at least one eigenvalue has a magnitude that is greater than or equal to 1, the discrete-time system is unstable. In this case, both of the eigenvalues are unstable.

(b) Suppose we use state feedback of the form $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] = F\vec{x}[i]$.

Find the appropriate state feedback constants, f_1, f_2 so that the state space representation of the resulting closed-loop system has eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

Solution: The closed loop system using state feedback has the form

$$\vec{x}[i+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i] \quad (65)$$

$$= \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \quad (66)$$

$$= \left(\begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] \quad (67)$$

Thus, the closed loop system has the form

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} -2+f_1 & 2+f_2 \\ -2+f_1 & 3+f_2 \end{bmatrix}}_{A_{cl}} \vec{x}[i] \quad (68)$$

Finding the characteristic polynomial of the above system, we have

$$\det \left(\lambda I - \begin{bmatrix} -2+f_1 & 2+f_2 \\ -2+f_1 & 3+f_2 \end{bmatrix} \right) = (\lambda + 2 - f_1)(\lambda - 3 - f_2) - (-2 - f_2)(2 - f_1) \quad (69)$$

$$= \lambda^2 - f_1\lambda - f_2\lambda - \lambda + f_1f_2 - 6 - 2f_2 + 3f_1 \quad (70)$$

$$- (-4 + f_1f_2 + 2f_1 - 2f_2) \quad (71)$$

$$= \lambda^2 - (1 + f_1 + f_2)\lambda + f_1 - 2 \quad (72)$$

However, we want to place the eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. That means we want

$$\lambda^2 - (1 + f_1 + f_2)\lambda + f_1 - 2 = \left(\lambda + \frac{1}{2}\right)\left(\lambda - \frac{1}{2}\right) \quad (73)$$

or equivalently:

$$\lambda^2 - (1 + f_1 + f_2)\lambda + f_1 - 2 = \lambda^2 - \frac{1}{4} \quad (74)$$

Equating the coefficients of the different powers of λ on both sides of the equation, we get,

$$1 + f_1 + f_2 = 0 \quad (75)$$

$$f_1 - 2 = -\frac{1}{4} \quad (76)$$

Solving the above system of equations gives us $f_1 = \frac{7}{4}, f_2 = -\frac{11}{4}$.

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