Homework 12

This homework is due on Saturday, April 22, 2023 at 11:59PM. Self-grades and HW Resubmissions are due the following Saturday, April 29, 2023 at 11:59PM.

1. **SVD Proofs**

   (a) **Show, for a general matrix** $A \in \mathbb{R}^{m \times n}$ **with** $\operatorname{rank}(A) = r$ **and** $A = U\Sigma V^\top$**, that** $\operatorname{Null}(A) = \operatorname{Col}(V_{n-r})$**.** *(HINT: How do we show two sets are equal? Try and use that approach here. Consider the outer product summation form for the SVD. Also, consider using the rank-nullity theorem that* $\dim \operatorname{Col}(A) + \dim \operatorname{Null}(A) = n$*.)*

   (b) **Show, for a general matrix** $A \in \mathbb{R}^{m \times n}$ **with** $\operatorname{rank}(A) = r$ **and** $A = U\Sigma V^\top$**, that** $\operatorname{Col}(A) = \operatorname{Col}(U_r)$**.**

   (c) **Show, for a general matrix** $A \in \mathbb{R}^{m \times n}$ **with** $\operatorname{rank}(A) = r$ **and** $A = U\Sigma V^\top$**, that** $\operatorname{Null}(A^\top) = \operatorname{Col}(U_{m-r})$ **and** $\operatorname{Col}(A^\top) = \operatorname{Col}(V_r)$**. Then show:**

      i. $\dim \operatorname{Col}(A) + \dim \operatorname{Null}(A^\top) = m$**,**

      ii. and $\operatorname{Col}(A)$ **and** $\operatorname{Null}(A^\top)$ **are orthogonal.**

2. SVD System ID

Previously, we saw instances for how to solve system ID problems when $D \in \mathbb{R}^{m \times n}$ is full rank (for $m > n$). Now, let us consider more generally the following problem of estimating $\vec{p}$ in

$$D\vec{p} = \vec{s} \tag{1}$$

where $\vec{p} \in \mathbb{R}^n$, $\vec{s} \in \mathbb{R}^m$, and $D \in \mathbb{R}^{m \times n}$. We assume that $\text{rank}(D) = r < \min(m, n)$, and we do not make any further assumptions on the relationship between m and n. Let's assume that D has an SVD given by

$$D = U\Sigma V^\top \tag{2}$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthonormal matrices. $\Sigma \in \mathbb{R}^{m \times n}$ has the following form:

$$\Sigma = \begin{bmatrix} \Sigma_r & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix} \tag{3}$$

where $\Sigma_r = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$ is a $r \times r$ diagonal matrix with nonzero elements along its diagonal.

Using this problem setup, we can rewrite our original system ID problem as

$$U\Sigma V^\top \vec{p} = \vec{s} \tag{4}$$

Our goal is to find $\vec{p}$ with smallest norm that best estimates $\vec{s}$.

For notational convenience, denote $U := \begin{bmatrix} U_r & U_{m-r} \end{bmatrix}$ where $U_r \in \mathbb{R}^{m \times r}$ is a matrix with the first r columns of U and $U_{m-r} \in \mathbb{R}^{m \times (m-r)}$ is a matrix with the last $m - r$ columns of U. Also, denote $V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$ where $V_r \in \mathbb{R}^{n \times r}$ is a matrix that has the first r columns of V, and $V_{n-r} \in \mathbb{R}^{n \times (n-r)}$ is a matrix that has the last $n - r$ columns of V. From SVD properties, we know that the columns of U_r form an orthonormal basis for $\text{Col}(D)$ and that the columns of V_{n-r} form an orthonormal basis for $\text{Null}(D)$.

(a) **Using the fact that U is orthonormal, show that $\Sigma V^\top \vec{p} = U^\top \vec{s}$.**

(b) **Show that we can write $\vec{p} = \begin{bmatrix} V_r & V_{n-r} \end{bmatrix} \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \end{bmatrix}$ for some vectors $\vec{\alpha}$ and $\vec{\beta}$ (i.e., find $\vec{\alpha} \in \mathbb{R}^r$ and $\vec{\beta} \in \mathbb{R}^{n-r}$). Show that changing $\vec{\beta}$ will not affect the result of $D\vec{p}$ and that we should set $\vec{\beta} = \vec{0}$ if we want to minimize $\|p\|$.** Since $V_{n-r}\vec{\beta}$ is the component of $\vec{p}$ that is in the nullspace of D, we can set $\vec{\beta}$ to be whatever we want. To choose the $\vec{p}$ with smallest norm, we will set $\vec{\beta} = \vec{0}$.

(c) From the previous part, we can rewrite $\vec{p} = V \begin{bmatrix} \vec{\alpha} \\ \vec{0} \end{bmatrix}$. This simplifies our system ID problem as follows:

$$\Sigma V^\top V \begin{bmatrix} \vec{\alpha} \\ \vec{0} \end{bmatrix} = U^\top \vec{s} \tag{5}$$

$$\Sigma \begin{bmatrix} \vec{\alpha} \\ \vec{0} \end{bmatrix} = U^\top \vec{s} \tag{6}$$

Simplify the left hand side of eq. (6) using eq. (3). Rewrite $U^\top \vec{s}$ as $\begin{bmatrix} U_r^\top \vec{s} \\ U_{m-r}^\top \vec{s} \end{bmatrix}$ and find an expression for $\vec{\alpha}$. *(HINT: Block matrix multiplication will work like normal matrix-vector multiplication here since $\Sigma_r \in \mathbb{R}^{r \times r}$ and $\vec{\alpha} \in \mathbb{R}^r$.)*

(d) **Use the previous part to come up with a solution for $\vec{p}$.**

(e) From the concept of projections, we know that the optimal solution for $\vec{p}$ satisfies the property that the projection error, namely $\vec{s} - D\vec{p}$, is orthogonal to the projection itself, namely $D\vec{p}$. Write $\vec{s} := \begin{bmatrix} U_r & U_{m-r} \end{bmatrix} \begin{bmatrix} \vec{w} \\ \vec{z} \end{bmatrix}$ for some vectors $\vec{w} \in \mathbb{R}^r$ and $\vec{z} \in \mathbb{R}^{m-r}$. **Find $\vec{w}$ and $\vec{z}$. Using this, show that our solution for $\vec{p}$ is optimal.**

3. Min Norm Proofs

Recall from the previous problem that we need to find a value of $\vec{x}_\star \in \mathbb{R}^n$ that best approximates

$$A\vec{x}_\star \approx \vec{y} \tag{7}$$

where $\vec{y} \in \mathbb{R}^m$. This is the typical problem of least squares, but sometimes we can have multiple values of $\vec{x}$ that approximate $A\vec{x} \approx \vec{y}$ equally well. To choose a unique solution, we pick the $\vec{x}_\star$ with minimum norm.

If A is rank $r = \text{rank}(A)$ and has SVD $A = U\Sigma V^\top$, we can write $U := \begin{bmatrix} U_r & U_{m-r} \end{bmatrix}$, $V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$, and $\Sigma = \begin{bmatrix} \Sigma_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$. From the previous homework, you determined that the optimal solution for $\vec{x}_\star$, given the requirements above, is

$$\vec{x}_\star = V \begin{bmatrix} \Sigma_r^{-1} U_r^\top \vec{y} \\ \vec{0}_{n-r} \end{bmatrix} \tag{8}$$

(a) The first property we will show is that $\vec{x}_\star \in \text{Col}\left(A^\top\right)$. **To do this, first prove that** $\text{Null}(A) \perp \text{Col}\left(A^\top\right)$**.** Use the fact that an SVD of $A^\top$ is $A^\top = V\Sigma U^\top$, and use Theorem 14 from Note 14. **Then, show that** $\dim \text{Null}(A) + \dim \text{Col}\left(A^\top\right) = n$**, and use this fact to argue that if a vector** $\vec{\ell} \perp \text{Null}(A)$ **(i.e., it is orthogonal to every vector in** $\text{Null}(A)$**), then** $\vec{\ell} \in \text{Col}\left(A^\top\right)$**.**

(HINT: When we are asked to show $\text{Null}(A) \perp \text{Col}\left(A^\top\right)$*, you need to argue that every vector in* $\text{Null}(A)$ *is orthogonal to every vector in* $\text{Col}\left(A^\top\right)$*.)*

(b) **Show that we can rewrite eq. (8) as**

$$\vec{x}_\star = V_r \Sigma_r^{-1} U_r^\top \vec{y} \tag{9}$$

Use this to show that $\vec{x}_\star \perp \text{Null}(A)$ **and hence** $\vec{x}_\star \in \text{Col}\left(A^\top\right)$**.**

(HINT: For the first part, write out $V = \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$ *and perform block matrix multiplication.) (HINT: For the second part, write* $\vec{x}_\star = V_r\vec{\alpha}$ *where* $\vec{\alpha} := \Sigma_r^{-1} U_r^\top \vec{y}$*. What does this mean about* $\vec{x}_\star$*'s relationship with the columns of* V_{n-r}*?)*

(c) Next, we will prove that, when $r = \text{rank}(A) = m$ (so A has to be a wide matrix), we have the following min norm solution:

$$\vec{x}_\star = A^\top \left(AA^\top\right)^{-1} \vec{y} \tag{10}$$

Using eq. (9), show that the above equation holds true. *(HINT: Use the compact SVD, namely* $A = U_r \Sigma_r V_r^\top$*.) (HINT:* U_r *should be a square, orthonormal matrix in this case. This is not necessarily the case for* V_r*, but remember that* $V_r^\top V_r = I$*.)*

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